Problèmes de contrôle stochastiques : contrôle sous contrainte, contrôlabilité et application à la réassurance

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Thèse

Problèmes de contrôle stochastiques : contrôle sous contrainte, contrôlabilité et application à la réassurance

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Sommaire

Introduction

Le but de cette thèse est de présenter quelques contributions dans le cadre du contrôle des équations différentielles stochastiques en dimension finie où infinie. Ce travail comporte quatre parties correspondant aux articles publiés ou soumis en vue de publication dans des revues mathématiques :

(1) Contrôle stochastique non borné sous contraintes d'état (Non-compact-valued stochastic control under state constraints)

(accepté à Bull. Sci. math (2006), doi :10.1016/ j.bulsci.2006.08.001) ;

Dans cette première partie, nous étudions une condition nécessaire sous laquelle les solutions d'une équation différentielle stochastique régie par un processus de contrôle non-borné restent dans un voisinage arbitrairement petit d'un ensemble donné de contraintes. On montre que, par rapport au problème classique de contrôle sous contraintes avec des processus de contrôle bornés, afin d'obtenir une condition nécessaire et suffisante de la viabilité en termes de solution de viscosité de l'équation de Hamilton–Jacobi–Bellman associée, on a besoin d'une hypothèse supplémentaire sur la croissance du processus de contrôle. Un exemple assez général illustre notre résultat principal.

(2) Contrôlabilité approchée pour des équations différentielles linéaires avec bruit contrôlé (Approximate controllability for linear SDEs with control acting on the noise) (publié dans Applied Analysis and Differential Equations, Iaşi, România 4 - 9 September 2006, World Scientific Publishing) ;

Dans cette deuxième partie, on s'intéresse à la propriété de contrôlabilité approchée pour une équation différentielle stochastique linéaire. Pour le contrôle déterministe, il existe une condition nécessaire et suffisante appelée condition de Kalman. Pour le cas stochastique, des critères sont connus soit dans le cas où le contrôle agit pleinement sur le bruit, soit dans le cas où il n'y a aucun contrôle sur le bruit. Nous proposons une généralisation de la condition de Kalman pour le cas général.

(3) Contrôlabilité approchée pour des équations différentielles linéaires en dimension infinie (Approximate Controllability for Linear Stochastic Differential Equations in Infinite Dimensions) ;

La troisième partie est dédiée à l'étude de la propriété de contrôlabilité approchée pour un système stochastique linéaire dans un espace de Hilbert réel et séparable. En particulier, nous montrons l'existence et unicité pour la solution de l'équation différentielle stochastique rétrograde duale associée au système initial. On doit souligner le fait que dans cette équation rétrograde les opérateurs qui agissent sur Y et Z sont non-bornés. On montre la dualité entre la propriété de contrôlabilité approchée pour le système initial et l'observabilité de l'équation duale. Dans le cas d'un générateur infinitésimal d'un semi-groupe exponentiellement stable, nous montrons que le test généralisé de Hautus donne une condition nécessaire pour la contrôlabilité approchée. Ceci généralise en partie les résultats obtenus dans la deuxième partie dans le cas fini-dimensionnel.

(4) Assurance, réassurance et paiement de dividendes (Insurance, Reinsurance and Dividend Payment) ;

Nous introduisons un modèle d'assurance qui permet la réassurance et le paiement des dividendes. Notre modèle prend en compte plusieurs contrats homogènes ainsi que la législation européenne en vigueur concernant les provisions des sociétés d'assurance. Ces éléments sont traduites par des restrictions sur le nombre (maximal) des contrats d'assurance. On travaille avec un processus de saut contrôlé avec libre choix sur le niveau de rétention et le montant des dividendes. La fonction valeur est donnée par la valeur maximisée des dividendes escomptés. Nous montrons que cette fonction est la solution de viscosité d'une inégalité variationnelle de Hamilton-Jacobi-Bellman de premier ordre. On obtient également un résultat d'unicité pour la solution de viscosité et nous discutons un exemple numérique.

Chapitre 1
Contrôle stochastique non borné sous contraintes d'état

1.1 Présentation du problème

On considère l'équation différentielle stochastique suivante

$$\begin{cases} dX^{x,v(\cdot)}(t) = b(X^{x,v(\cdot)}(t), v(t))dt + \sigma(X^{x,v(\cdot)}(t), v(t))dW(t),\ t \geq 0, \\ X^{x,v(\cdot)}(0) = x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

gouvernée par un processus de contrôle $v(\cdot)$ à valeurs dans un espace d'état de contrôle non-borné. Etant donné un sous-ensemble fermé K de $\mathbb{R}^n$, on souhaite obtenir une condition nécessaire et suffisante sur les coefficients de l'équation sous laquelle, pour chaque point de départ $x \in \mathbb{R}^n$, on trouve un processus admissible de contrôle tel que la trajectoire associée reste dans l'ensemble K (ou, au moins dans un voisinage arbitrairement petit de cet ensemble). Cette propriété est appelée viabilité (ou ε-viabilité) et elle a été beaucoup étudiée pour les systèmes déterministes ainsi que les systèmes stochastiques.

Dans le cadre déterministe, la propriété de viabilité d'un ensemble fermé a été étudiée pour la première fois par Nagumo en 1943. Aubin et Da Prato [3] et [4], ainsi que Gauthier et Thibault [25], ont étendu le théorème de viabilité de Nagumo au cas des équations différentielles stochastiques. Le point clef dans leur travail est l'utilisation du "cône tangent stochastique" qui généralise le cône contingent de Bouligand utilisé dans le cadre déterministe. Le théorème de viabilité stochastique a été obtenu dans le cas des coefficients continus ainsi que pour des inclusions différentielles stochastiques dont le terme de droite est de type Marchaud par Aubin et Da Prato [4].

Dans le cas des systèmes stochastiques contrôlés dont l'espace de contrôle est compact, une approche qui utilise la notion de solution au sens de viscosité des EDP de second ordre a été développée par Buckdahn, Peng, Quincampoix et Rainer [9]. Leur résultat principal donne comme condition nécessaire et suffisante pour la viabilité du fermé K la condition que le carré de la fonction distance de K soit une sursolution de viscosité de l'équation de Hamilton-Jacobi-Bellman associée. Dans Buckdahn, Quincampoix, Rainer, Rascanu [12], la même approche est utilisée pour l'étude de la propriété de viabilité des ensembles dépendant du temps ; le système stochastique considéré ici est non contrôlé. Dans Buckdahn, Cardaliaguet et Quincampoix [11], les résultats obtenus ont été étendus aux systèmes stochastiques contrôlés. Plus récemment, Buckdahn, Quincampoix et Tessitore [14] ont utilisé cette méthode pour obtenir une caractérisation de la viabilité par rapport à des systèmes stochastiques infini dimensionnels contrôlés.

Michta [36] considère un problème de viabilité où le choix est restreint sur la condition initiale. Il étudie la propriété de "viabilité faible" (la condition de viabilité doit être satisfaite à chaque instant t avec une probabilité suffisamment grande). Le même type de viabilité est étudie par Mazliak [34].

Le Théorème de Filippov pour les inclusions différentielles a été étendu au cas de inclusions différentielles stochastique dont le terme de droite a la propriété de Lipschitz par Da Prato et Frankowska [17]. Ce résultat a été utilisé par Aubin, Da Prato et Frankowska [5] pour caractériser l'invariance d'un ensemble par rapport à une inclusion différentielle stochastique ayant la propriété de Lipschitz. Bardi et Goatin [7] ont donné une caractérisation de l'invariance en utilisant les “cônes normaux de second ordre” et des techniques de solutions de viscosité pour des équations de Hamilton-Jacobi de second ordre.

Dans le contexte des solutions faibles des systèmes stochastiques contrôlés, on rappelle le travail de Da Prato, Frankowska [18] dont la méthode emploie une généralisation de la transformée de Doss-Sussman.

Dans cette partie on utilise une approche basée sur la notion de solution de viscosité. Plus précisément, on montre que la propriété de ε-viabilité de l'ensemble K est équivalente au fait que la fonction valeur

$$V(x) = \inf E\left[\int_0^\infty e^{-Cs} d_K^2(X^{x,v(\cdot)}(s))ds\right],\ x \in \mathbb{R}^n, \tag{1.2}$$

soit une sursolution de viscosité de l'équation de Hamilton-Jacobi-Bellman associée au système stochastique contrôlé.

Dans le cas d'un espace d'état de contrôle U compact (cf. [9]), afin de caractériser la propriété de ε-viabilité d'un ensemble fermé $K \subset \mathbb{R}^n$, les auteurs de [9] ont introduit la fonction valeur suivante

$$V(x) = \inf_{v \in \mathcal{A}} E\left[\int_0^\infty e^{-Cs} d_K^2(X^{x,v(\cdot)}(s))ds\right]. \tag{1.3}$$

ainsi que l'équation de Hamilton-Jacobi-Bellman associée

$$\begin{aligned} 0 \ &= \ \sup_{v \in U}\left\{ - < D_x u(x) b(x,v) > - \frac{1}{2} Tr[D_{xx}^2 u(x) \sigma\sigma^*(x,v)]\right\} \\ &\quad + Cu(x) - d_K^2(x),\ x \in \mathbb{R}^n. \end{aligned} \tag{1.4}$$

Leur résultat peut être résumé par

Théorème 1 *([9], Th. 2) Sous les hypothèses*

i) les fonctions $b : \mathbb{R}^n \times U \longrightarrow \mathbb{R}^n$ et $\sigma : \mathbb{R}^n \times U \longrightarrow \mathbb{R}^{n\times d}$ ont une croissance au plus linéaire,

ii) les coefficients b, σ sont uniformément continus sur $\mathbb{R}^n \times U$ et ont la propriété de Lipschitz en $x \in \mathbb{R}^n$, uniformément en $u \in U$,

les assertions suivantes sont équivalentes :

a) pour tout $x \in K$, $V(x) = 0$;

b) la fonction d_K^2 est une sursolution de viscosité de (1.4).

Dans le cas d'un espace d'état de contrôle non-borné, nous montrons que, en général, la propriété de ε-viabilité n'implique pas nécessairement une condition de type sursolution de viscosité. En effet, on peut montrer un exemple pour $K = \{0, 1\}$ où, même si la fonction valeur V s'annule sur K, pour une suite de processus de contrôle bien choisie, la solution parcourt tout l'intervalle $(-1, 0)$, ce qui est loin de la propriété de viabilité. Cela peut être évité si on introduit une équation supplémentaire agissant comme terme régulateur, équation interprétée comme coût lié au contrôle. Plus précisément, on étudie le système contrôlé suivant

$$\begin{cases} dX^{x,v(\cdot)}(t) = b(X^{x,v(\cdot)}(t), v(t))dt + \sigma(X^{x,v(\cdot)}(t), v(t))dW(t); \\ dY^{x,y,v(\cdot)}(t) = f(X^{x,v(\cdot)}(t), Y^{x,y,v(\cdot)}(t), v(t))dt,\ t \geq 0; \\ X^{x,v(\cdot)}(0) = x \in \mathbb{R}^n;\ Y^{x,y,v(\cdot)}(0) = y \in \mathbb{R}, \end{cases} \tag{1.5}$$

où $f(x, y, v) \geq |v|^p - \beta(x)$, et β est une fonction continue.

1.2 Les résultats

1.2.1 Contre-exemple dans le cas non-borné

L'exemple suivant montre que, dans le cas non-borné, sans hypothèse supplémentaire, on peut trouver un ensemble K tel que la fonction valeur s'annule sur K mais la fonction distance carré d_K^2 ne soit pas une sursolution de viscosité de l'équation de Hamilton-Jacobi-Bellman associée.

Exemple 1 *On considère* $d = n = k = 1$, $\sigma : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, *avec* $\sigma(x, v) = 0$, $x, v \in \mathbb{R}$, *et* $b : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, *avec* $b(x, v) = |x| + |v|$, $x, v \in \mathbb{R}$. *L'espace des processus de contrôle* $\mathcal{A} = L^2_{loc}(\mathbb{R}, dt)$. *On étudie la viabilité de l'ensemble* $K = \{-1, 0\} \subset \mathbb{R}$. *Pour cela, on définit*

$$V(x) = \inf_{u \in \mathcal{A}} \int_0^\infty e^{-Cs} d_K^2(X^{x,u(\cdot)}(s)) ds.$$

On montre facilement que $V(0) = 0$. *En considérant les processus de contrôle*

$$u^v(t) = \begin{cases} v, \ \textit{if}\, t \in [0, \ln(\frac{1+v}{v})); \\ 0, \ \textit{if}\, t \geq \ln(\frac{1+v}{v}), \end{cases}$$

on montre que $V(-1) = 0$ *mais la solution* $X^{-1,u^v(\cdot)}$ *parcourt l'intervalle* $[-1, 0)$ *et* d_K^2 *n'est pas une sursolution de viscosité de l'équation de Hamilton-Jacobi-Bellman associée.*

1.2.2 Contrôle stochastique dans L^p

La famille de processus de contrôle $\{u^v(\cdot),\ v > 0\}$est bornée dans $L^1(\mathbb{R}, e^{-Cs}ds)$. Néanmoins, pour tout $p > 1$, la famille $\{u^v(\cdot),\ v > 0\}$ n'est plus bornée dans $L^p(\mathbb{R}, e^{-Cs}ds)$. Ainsi, il est naturel d'introduire une équation additionnelle qui permettra d'obtenir une condition sur les processus de contrôle optimaux (où ε-optimaux)

$$\left\{\begin{array}{l} dX^{x,v(\cdot)}(t) = b(X^{x,v(\cdot)}(t), v(t))dt + \sigma(X^{x,v(\cdot)}(t), v(t))dW(t), \\ dY^{x,y,v(\cdot)}(t) = f(X^{x,v(\cdot)}(t), Y^{x,y,v(\cdot)}(t), v(t))dt,\ t \geq 0, \\ X^{x,v(\cdot)}(0) = x \in \mathbb{R}^n;\ Y^{x,y,v(\cdot)}(0) = y \in \mathbb{R}. \end{array}\right. \tag{1.6}$$

On se place sous les hypothèses suivantes

(A.1) *Il existe $L > 0$ tel que les coefficients $b : \mathbb{R}^n \times \mathbb{R}^k \longrightarrow \mathbb{R}^n$, $\sigma : \mathbb{R}^n \times \mathbb{R}^k \longrightarrow \mathbb{R}^{n\times d}$, $f : \mathbb{R} \times \mathbb{R}^k \longrightarrow \mathbb{R}$ satisfont*

$$\begin{array}{rcl} |b(x,u) - b(x',u)| & \leq & L|x - x'| \\ |\sigma(x,u) - \sigma(x',u)| & \leq & L|x - x'| \\ |f(x,y,u) - f(x,y',u)| & \leq & L(|y - y'| + |x - x'|) \end{array}$$

pour tous $x, x' \in \mathbb{R}^n$, $y, y' \in \mathbb{R}$, $u \in \mathbb{R}^k$.

(A.2) *b et σ sont uniformément continus sur $\mathbb{R}^n \times \mathbb{R}^k$ et il existe $L > 0$ tel que*

$$\begin{array}{rcl} |b(x,u)| & \leq & L(1 + |x| + |u|) \\ |\sigma(x,u)| & \leq & L(1 + |x| + |u|) \end{array}$$

pour tous $x \in \mathbb{R}^n$, $u \in \mathbb{R}^k$.

La fonction f est uniformément continue, $\sup_{x\in\mathbb{R}^n, y\in\mathbb{R}} |f(x,y,0)| \leq L$, et, pour tous $x \in \mathbb{R}^n$ et $u \in \mathbb{R}^k$, $f(x,\cdot,u)$ est positive sur $[l,\infty)$. En plus, il existe $\beta : \mathbb{R}^n \to \mathbb{R}$, continue, telle que

$$f(x,y,u) \geq |u|^p - \beta(x),$$

pour tous $x \in \mathbb{R}^n$, $y \in \mathbb{R}$, $u \in \mathbb{R}^k$.

On désigne par $\mathcal{A}$ la famille des processus de contrôle admissibles, i.e. la famille des processus progressivement mesurables $v(\cdot)$ à valeurs dans $\mathbb{R}^k$ tels que

$$E\left[\int_0^T |v(s)|^p ds\right] < \infty, \text{ pour tous } T > 0.$$

On considère un ensemble fermé $K \subset \mathbb{R}^n$ et désignons par B_l l'intervalle $[-l, l] = \{y \in \mathbb{R}, |y| \leq l\}$, où $l > 0$ est fixé.

(A.3)

$$f(x, y, v) = f(\pi_K(x), y, v),$$

pour une sélection mesurable $\pi_K : \mathbb{R}^n \longrightarrow K$, *telle que* $\pi_K(x) \in \Pi_K(x) = \{y \in K : d_K(x) = |x - y|\}$ *pour tous* $x \in \mathbb{R}^n$, $y \geq l$, $u \in \mathbb{R}^k$.

Pour tout processus de contrôle admissible $v(\cdot)$ et tous $(x, y) \in \mathbb{R}^n \times \mathbb{R}$ on introduit la fonctionnelle de coût

$$J(x, y; v(\cdot)) = E\left[\int_0^\infty e^{-Cs} \left(d_K^2(X^{x,v(\cdot)}(s) \wedge 1) + d_{B_l}(Y^{x,y,v(\cdot)}(s))\right) ds\right] \tag{1.7}$$

pour $C > 0$ assez grand, et

$$V(x, y) = \inf_{v \in \mathcal{A}} J(x, y; v(\cdot)). \tag{1.8}$$

On obtient la régularité de la fonction valeur

Proposition 1 *La fonction V a la propriété de Lipschitz.*

On montre facilement (cf [23]) que V est l'unique solution de viscosité dans la classe des fonctions de croissance au plus quadratique pour l'équation de Hamilton-Jacobi-

Bellman associée

$$0 = \sup_{v\in\mathbb{R}^k}\{- < D_xV(x,y), b(x,v) > -D_yV(x,y)f(x,y,v) \\ -\frac{1}{2}Tr[D^2_{xx}V(x,y)\sigma\sigma^*(x,v)]\} + CV(x,y) - d^2_K(x)\wedge 1 - d_{B_l}(y)\ . \quad (1.9)$$

Motivé par les résultats de [9], on introduit $F : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}$ définie par $F(x,y) = d^2_K(x)\wedge 1 + d_{B_l}(y)$. Notre théorème principal est

Théorème 2 *Sous les hypothèses (A.1)-(A.3), les assertions suivantes sont équivalentes :*

(i) L'ensemble $K\times B_l$ a la propriété de ε-viabilité pour (1.6);

(ii) La fonction $F : \mathbb{R}^n\times\mathbb{R} \longrightarrow \mathbb{R}$, $F(x,y) = d^2_K(x)\wedge 1 + d_{B_l}(y)$, $(x,y)\in\mathbb{R}^n\times\mathbb{R}$, est une sursolution de viscosité pour (1.4).

1.2.3 Un exemple

Exemple 2 *On pose $\widehat{\pi} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ la projection sur les $n-1$ premières coordonnées*

$$\widehat{\pi}(x) = (x_1, x_2, \cdots, x_{n-1}, 0),\ \textit{pour}\ x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$$

et $\pi_n : \mathbb{R}^n \longrightarrow \mathbb{R}$ la projection sur la dernière coordonnée, $\pi_n((x_1, x_2, \cdots, x_n)) = x_n$.

On étudie la viabilité du cylindre $K = \{x\in\mathbb{R}^n : |\widehat{\pi}(x)| \le R\}$.

Proposition 2 *Si $|\sigma(\cdot,v)\widehat{I}|^2$ est Lipschitz uniformément en $v\in\mathbb{R}^k$ (où $\widehat{I} = \begin{pmatrix} I_{n-1} & 0 \\ 0 & 0\end{pmatrix}$ et I_{n-1} est la matrice unité dans $\mathbb{R}^{(n-1)\times(n-1)}$), alors K a la propriété de ε-viabilité pour l'équation (1.6) si et seulement si les conditions suivantes sont satisfaites simultanément :*

(a) Pour tout $x \in \mathbb{R}^n$ *tel que* $|\widehat{\pi}(x)| = R$*, il existe* $v \in \mathbb{R}^k$ *tel que*

$$\begin{cases} (1)\ \sigma^*(x,v)\widehat{\pi}(x) = 0; \\ (2)\ 2 < \widehat{\pi}(x), b(x,v) > + |\sigma^*(x,v)\widehat{I}|^2 \leq 0; \\ (3)\ f(x,l,v) = 0. \end{cases}$$

(b) Pour tout $x \in \mathbb{R}^n$ *tel que* $|\widehat{\pi}(x)| < R$*, il existe* $v \in \mathbb{R}^k$ *tel que*

$$f(x,l,v) = 0.$$

Chapitre 2
Contrôlabilité approchée pour des équations différentielles linéaires avec bruit contrôlé

2.1 Présentation du problème

Nous étudions la contrôlabilité approchée pour l'équation différentielle stochastique linéaire

$$\begin{cases} dy(t) = (Ay(t) + Bu(t))\,dt + (Cy(t) + Du(t))\,dW(t),\ 0 \le t \le T, \\ y(0) = x \in \mathbb{R}^n, \end{cases} \tag{2.10}$$

gouvernée par un processus de contrôle $u(\cdot)$, où $A, C \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, et $B, D \in \mathcal{L}(\mathbb{R}^d, \mathbb{R}^n)$.

On rappelle la définition de la contrôlabilité approchée et de la 0-contrôlabilité approchée

Définition 1 *L'équation (*2.10*) a la propriété de contrôlabilité approchée si, pour tout $x \in \mathbb{R}^n$, tout $T > 0$, tout $\eta \in L^2(\Omega\,; \mathcal{F}_T\,; P\,; \mathbb{R}^n)$, tout $\varepsilon > 0$, il existe un contrôle admissible u tel que*

$$E\left[|y(T, x, u) - \eta|^2\right] \le \varepsilon.$$

*L'équation (*2.10*) a la propriété de 0-contrôlabilité approchée si la condition ci-dessus a lieu pour $\eta = 0$.*

Pour le cas où le coefficient du processus de contrôle D est de rang plein, l'auteur de [39] a montré que la propriété de contrôlabilité exacte pour (2.10) peut être caractérisée à l'aide de conditions algébriques de type Kalman. Si la matrice D n'est pas de rang plein, l'équation (2.10) n'est pas exactement contrôlable. Dans ce cas on étudie la contrôlabilité approchée. Cette propriété a été étudiée dans [13] pour le cas spécial $D = 0$. Les auteurs généralisent la condition de Kalman pour obtenir un critère équivalent pour la contrôlabilité approchée de (2.10).

Nous proposons une extension de ces résultats au cas général où le contrôle peut agir également sur le bruit (i.e. $rang\ D \geq 0$) sans forcément avoir D de rang plein. Plus précisément, nous montrons l'équivalence entre la propriété de contrôlabilité approchée, celle de 0-contrôlabilité approchée et une notion d'invariance conditionnelle. Cette dernière notion est facilement calculable.

2.2 Les résultats

Il suffit de réduire l'étude à l'équation suivante, qui est équivalente à (2.10)

$$dy(t) = (Ay(t) + B_1u'(t) + B_2u''(t))dt + (Cy(t) + D_1u'(t))dW(t), \tag{2.11}$$

où $A, C \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, $B_1, D_1 \in \mathcal{L}(\mathbb{R}^r, \mathbb{R}^n)$, $B_2 \in \mathcal{L}(\mathbb{R}^{d-r}, \mathbb{R}^n)$ et $rang\ D_1 = rang\ D = r$. En utilisant $rang\ D_1 = r$ on peut trouver $F \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ solution de $D_1^*F + B_1^* = 0$.

Nous utilisons la définition suivante

Définition 2 *Etant donnés les opérateurs linéaires $L, M, N \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ et les sous-espaces vectoriels $V \subset \mathbb{R}^n$ et $U \subset \mathbb{R}^n$, on dit que V est $(L; M)$-strictement invariant conditionnellement à (N, U) si pour tout $v \in V$ il existe $w \in V$ tel que*

$$w - Nv \in U \text{ et } Lv + Mw \in V.$$

Remarque 1 *Etant donnés les sous-espaces vectoriels* $V, U \subset \mathbb{R}^n$*, le plus grand sous-espace de* V *qui est* $(L; M)$*-strictement invariant conditionnellement à* (N, U) *peut être obtenu en considérant le schéma itératif suivant :*

$$V_0 = V;\ V_{i+1} = \{v \in V_i :\ M((U + Nv) \cap V_i) \cap (V_i - Lv) \neq \phi\}\,,\ i \in \mathbb{N}.$$

2.2.1 L'équation différentielle stochastique rétrograde duale

Le premier résultat concerne la dualité qui existe entre la propriété de contrôlabilité approchée pour l'équation stochastique linéaire et une notion d'observabilité pour l'équation rétrograde associée. Pour cela, on considère le système

$$\begin{cases} dp(t) = [-(A^* + C^*F)p(t) - C^*q(t)]\,dt + (Fp(t) + q(t))dW(t), \\ p(T) = \eta. \end{cases} \tag{2.12}$$

La dualité est donnée par

Proposition 3 *L'équation (*2.11*) est approximativement-contrôlable si et seulement si, pour tout* $T > 0$*, chaque solution de (*2.12*) vérifiant* $B_2^* p(s) = 0$ *et* $D_1^* q(s) = 0$*,* $P - p.s.$*, pour tout* $s \in [0, T]$ *est réduite à* 0*.*

*En plus, l'équation (*2.11*) est approximativement 0-contrôlable si et seulement si, pour tout* $T > 0$*, chaque solution de (*2.12*) vérifiant* $B_2^* p(s) = 0$ *et* $D_1^* q(s) = 0$*,* $P - p.s.$*, pour tout* $s \in [0, T]$ *satisfait* $p(0) = 0$*.*

On peut voir l'équation (2.12) comme un système contrôlé

$$\begin{cases} dp(t, q, \theta) = [-(A^* + C^*F)p(t, q, \theta) - C^*q(t)]\,dt + (Fp(t, q, \theta) + q(t))dW(t); \\ p(0, q, \theta) = \theta \in \mathbb{R}^n. \end{cases} \tag{2.13}$$

Dans ce cas, la proposition précédente s'écrit

Proposition 4 *L'équation (*2.11*) est approximativement-contrôlable si et seulement si, pour tous $T > 0$, $\theta \in \mathbb{R}^n$, et tout $q \in L^2_{\mathcal{P}}([0,T], Ker\ D_1^*)$ tel que $B_2^* p(s,q,\theta) = 0$, $P-p.s.$, pour tout $s \in [0,T]$, on a $q(s) = 0$, $dsdP$-presque partout sur $[0,T] \times \Omega$ et $\theta = 0$.*

*L'équation (*2.11*) est approximativement 0-contrôlable si et seulement si, pour tous $T > 0$, $\theta \in \mathbb{R}^n$ et tout $q \in L^2_{\mathcal{P}}([0,T], Ker\ D_1^*)$ tel que $B_2^* p(s,q,\theta) = 0$, $P-p.s.$, pour tout $s \in [0,T]$, on a $\theta = 0$.*

2.2.2 Viabilité conditionnelle

Pour obtenir un critère calculable pour la contrôlabilité approchée, les auteurs de [13] ont utilisé la notion de "viabilité locale en temps". Motivés par la proposition précédente et par cette approche, on introduit la notion de viabilité locale en temps conditionnelle.

Définition 3 *Soit $U, V \subset \mathbb{R}^n$ deux sous-espaces vectoriels de $\mathbb{R}^n$. La famille de tous les $\theta \in V$ pour lesquels il existe $T > 0$ et $q \in L^2_{\mathcal{P}}([0,T], U)$ tels que $p(s,q,\theta) \in V$, $P-p.s.$, pour tout $s \in [0,T]$ est appellée noyau de viabilité de V conditionnellement à U par rapport à l'équation (*2.13*) (on désigne cet ensemble par $Viab(V/U)$).*

*En plus, on dit que V est localement viable en temps conditionnellement à U par rapport à (*2.13*) si $Viab(V/U) = V$.*

Si U et V sont deux sous-espaces vectoriels de $\mathbb{R}^n$, on introduit $\Pi_{U^\perp}$ (respectivement $\Pi_{V^\perp}$) les projecteurs orthogonaux sur $U^\perp$ (respectivement $V^\perp$). Pour $N \geq 1$, on considère l'équation de Riccati à valeurs dans $\mathcal{S}^n$ (la famille des matrices symétriques, positive de type $n \times n$) suivante :

$$\begin{cases} P_N'(s) = -P_N(s)(A^* + C^*F) - (A + F^*C)P_N(s) + F^*P_N(s)F \\ \qquad - (F^*P_N(s) - P_N(s)C^*)(I + N\Pi_{U^\perp} + P_N(s))^{-1}(P_N(s)F - CP_N(s)) \\ \qquad + N\Pi_{V^\perp}, \\ P_N(T) = 0; \end{cases} \tag{2.14}$$

Comme $I + N\Pi_{U^\perp} \gg 0$ et $N\Pi_{V^\perp} \geq 0$ (i.e. $N\Pi_{V^\perp}$ est positive), l'équation de Riccati (2.14) admet une unique solution à valeurs dans $\mathcal{S}^n$ (cf [47], condition (4.23) et Théorème 7.2).

Cette équation nous permet d'obtenir une caractérisation explicite du noyau de viabilité conditionnelle

Proposition 5 *Le noyau de viabilité de V conditionnellement à U par rapport à (*2.13*) est donné par*

$$Viab(V|U) = \left\{\theta \in V \,:\, \exists T > 0 \; s.t.. \lim_{N\to\infty} \langle P_N(T)\theta, \theta\rangle < \infty \right\}.$$

Ainsi, en utilisant ce résultat, on montre facilement

Proposition 6 *Le noyau de viabilité d'un sous-espace vectoriel $V \subset \mathbb{R}^n$ conditionnellement au sous-espace vectoriel $U \subset \mathbb{R}^n$ par rapport à (*2.13*) est conditionnellement localement viable en temps. En particulier, le noyau de viabilité conditionnelle est localement viable en temps.*

2.2.3 Le résultat principal

On obtient un critère permettant d'exprimer la contrôlabilité approchée à l'aide du noyau de viabilité

Théorème 3 *Les assertions suivantes sont équivalentes :*

*(1) L'équation (*2.11*) est approximativement-contrôlable.*

*(2) L'équation (*2.11*) est approximativement 0- contrôlable.*

(3) Le noyau de viabilité de $Ker\ B_2^*$ *conditionnellement au* $Ker\ D_1^*$ *est trivial.*

Il reste encore à donner une condition facilement calculable équivalente à la condition (3). En effet

Proposition 7 *Le sous-espace vectoriel* $V \subset \mathbb{R}^n$ *est localement viable en temps conditionnellement au sous-espace vectoriel* $U \subset \mathbb{R}^n$ *par rapport à (*2.13*) si et seulement si* V *est* $(A^*; C^*)$*-strictement invariant conditionnellement à* (F, U).

Ainsi, on obtient notre résultat principal

Théorème 4 *Les assertions suivantes sont équivalentes :*

*(1) L'équation (*2.11*) est approximativement-contrôlable.*

*(2) L'équation (*2.11*) est approximativement 0- contrôlable.*

(3) Le plus grand sous-espace vectoriel de $KerB_2^*$ *qui est* $(A^*; C^*)$*-strictement invariant conditionnellement à* $(F, KerD_1^*)$ *est le sous-espace* $\{0\}$.

Cette dernière condition est facilement calculable (voir remarque 1).

Chapitre 3
Contrôlabilité approchée pour des équations différentielles linéaires en dimension infinie

3.1 Présentation du problème

Nous nous intéressons à l'étude de la contrôlabilité approchée pour une équation différentielle stochastique en dimension infinie avec bruit multiplicatif du type

$$\begin{cases} dX_t^{x,u} = \left(AX_t^{x,u} + Bu_t\right) dt + CX_t^{x,u} dW_t,\ t \geq 0. \\ X_0 = x \in H, \end{cases} \tag{3.15}$$

où u est un processus de contrôle à valeurs dans U et l'espace d'état H ainsi que l'espace de contrôle U sont des espace de Hilbert réels et séparables.

Définition 4 *L'équation (*3.15*) a la propriété de contrôlabilité approchée (au temps $T > 0$) si et seulement si, pour tout $x \in H$, tout $\varepsilon > 0$ et toute condition finale $\eta \in L^2\left(\Omega, \mathcal{F}_T, P; H\right)$ il existe un processus de contrôle u_ε tel que la trajectoire partant de x et associée au processus de contrôle u_ε vérifie*

$$E\left[\left|X_T^{x,u_\varepsilon} - \eta\right|^2\right] \leq \varepsilon.$$

Pour les systèmes contrôlés en dimension finie, la propriété de contrôlabilité est complètement caractérisée par la condition de Kalman. Parfois, au lieu d'étudier la contrôlabilité du système initial, on étudie la propriété d'observabilité pour le système dual associé. Pour cela, un outil très important le constitue le test d'Hautus. Dans le cas des systèmes ayant comme espace d'état un espace de Hilbert, sous certaines conditions, Russell et Weiss

[42] ont obtenu une condition nécessaire d'observabilité, condition qui généralise le test d'Hautus dans le cas fini-dimensionnel. Jacob et Zwart [29] ont montré que la condition de Russell et Weiss est un critère nécessaire et suffisant pour l'observabilité des systèmes diagonaux fortement stable dans le cas où l'espace de contrôle est fini-dimensionnel.

Pour les systèmes linéaires stochastique contrôlés dont l'espace d'état est fini-dimensionnel, Buckdahn, Quincampoix et Tessitore [13] ont obtenu un critère de type Kalman équivalent à la contrôlabilité approchée dans le cas où le coefficient de diffusion n'est pas contrôlé. Ce résultat a été généralisé par Goreac [26] au cas d'un bruit contrôlé (voir également Chapitre 2).

On souhaite généraliser l'approche utilisée dans le cas fini-dimensionnel aux systèmes contrôlés dont l'espace d'état est un espace de Hilbert réel et séparable. La première difficulté réside dans le fait que l'équation rétrograde associée peut ne pas être bien posée. Pour résoudre ce problème, on doit imposer une condition de dissipativité jointe (ce qui correspond a une propriété d'ellipticité déjà classique pour une équation de la chaleur, par exemple). L'équation duale ainsi obtenue est interprétée, dans le cas fini-dimensionnel, comme une équation directe contrôlée et la suite du démarche est basée sur les méthodes de Riccati algébriques. Cependant, dans le cas d'un système infini-dimensionnel, l'équation duale admet seulement une solution au sens mild ce qui nous empêche de la considérer comme équation directe contrôlée. Ainsi, une fois la propriété de contrôlabilité réduite a l'observabilité du système duale, on essaie d'étudier plutôt cette propriété duale à l'aide des conditions de type Kalman.

3.2 Les résultats

3.2.1 Introduction

On considère $(H, \langle\cdot,\cdot\rangle_H)$, $(U, \langle\cdot,\cdot\rangle_U)$, $(\Xi, \langle\cdot,\cdot\rangle_\Xi)$ des espaces de Hilbert réels et séparables. On introduit $\mathcal{L}(\Xi, H)$, l'espace des opérateurs linéaires bornés de Ξ à valeurs dans H et $L_2(\Xi, H)$ le sous-espace des opérateurs Hilbert-Schmidt avec leur normes usuelles. On se place sous les hypothèses suivantes

(A.0) *1) l'opérateur linéaire $A : D(A) \subset H \longrightarrow H$ est le générateur d'un C_0-semigroupe,*

2) l'opérateur $B \in \mathcal{L}(U; H)$,

3) l'operateur linéaire $C : H \longrightarrow \mathcal{L}(\Xi, H)$ est tel qu'il existe $\gamma \in \left[0, \frac{1}{2}\right)$ et $L > 0$ tels que, pour tout $t > 0$,

$$a)\; e^{tA}C \in \mathcal{L}\left(H; L_2(\Xi, H)\right),$$

$$b)\; \left|e^{tA}C\right|_{\mathcal{L}(H;L_2(\Xi,H))} \leq Lt^{-\gamma},$$

4) l'espace $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ est un espace de probabilité complet muni d'une filtration qui satisfait les conditions usuelles,

5) le processus W est un processus de Wiener cylindrique ($\mathcal{F}_t$)$-$adapté à valeurs dans Ξ ;

(A.1) *L'opérateur C peut être écrit comme somme de deux opérateurs linéaires C_1, C_2*

$$C = C_1 + C_2,$$

tels que :

1) C_2 est borné de H sur $L_2(\Xi; H)$,

2) pour tout $t > 0$, $C_1 e^{tA} \in \mathcal{L}(H; L_2(\Xi; H))$. On suppose également qu'il existe $\gamma \in \left[0, \frac{1}{2}\right)$ et $L > 0$ tels que

$$\left|C_1 e^{tA}\right|_{\mathcal{L}(H;L_2(\Xi;H))} \leq L t^{-\gamma},$$

pour tout $t > 0$.

3) il existe $a > \frac{1}{2}$ tel que

$$A + a\left(C_1 e^{\delta A}\right)^* \left(C_1 e^{\delta A}\right) \text{ est dissipatif,}$$

pour une suite $\delta \searrow 0$.

Si C_1 n'est pas l'opérateur trivial 0, on suppose

(A.2) $-A^2$ *est dissipatif.*

Remarque 2 *Si A est un opérateur linéaire dissipatif, auto-adjoint qui est le générateur d'un semigroupe de contractions, alors (A.2) est vérifiée.*

Si on suppose, de plus, que C_1 prend ses valeurs dans $L_2(\Xi; H)$, alors (A.1) 3 peut être remplacée par

3') il existe $a > \frac{1}{2}$ tel que

$$A + aC_1^* C_1 \textit{ est dissipatif.}$$

3.2.2 L'équation duale

On considère le système dual

$$\begin{cases} dY_t = -\left(A^*Y_t + C^*Z_t\right)dt + Z_t dW_t, \\ Y_T = \xi \in L^2\left(\Omega, \mathcal{F}_T, P; H\right). \end{cases} \tag{3.16}$$

Définition 5 *On appelle solution au sens "mild" de l'équation (*3.16*) un couple* (Y, Z) *de processus progressivement mesurables à valeurs dans* H*, respectivement* $L_2(\Xi, H)$*, tel que*

$$\begin{aligned} (Y,Z) \in\ & C\left([0,T]\,; L^2\left(\Omega; H\right)\right) \times L^2\left([0,T] \times \Omega; L_2\left(\Xi; H\right)\right), \\ & \sup_{t\in[0,T]} E\left[\left|Y_t\right|^2\right] + E\left[\int_0^T \left|Z_t\right|^2 dt\right] < \infty, \\ & \int_0^T \left|\left(Ce^{(s-t)A}\right)^* Z_s\right| ds < \infty,\ P-a.s.,\ et \\ & Y_t = e^{(T-t)A^*}\xi + \int_t^T \left(Ce^{(s-t)A}\right)^* Z_s ds - \int_t^T e^{(s-t)A^*} Z_s dW_s,\ t \in [0,T]\,. \end{aligned}$$

Le premier résultat concerne l'existence et l'unicité de la solution au sens "mild" de l'équation duale (3.16) :

Théorème 5 *Sous les hypothèses A.1 et A.2, il existe une unique solution de l'équation différentielle stochastique rétrograde linéaire (*3.16*). En plus, cette solution satisfait*

$$\sup_{t\in[0,T]} E\left[\left|Y_t\right|^2\right] + E\left[\int_0^T \left|Z_s\right|^2 ds\right] \leq kE\left[\left|\xi\right|^2\right], \tag{3.17}$$

pour un $k > 0$.

Le lien entre la propriété de contrôlabilité approchée pour l'équation directe (3.15) et l'observabilité du système dual (3.16) est donné par

Proposition 8 *L'équation (*2.11*) est approximativement-contrôlable si et seulement si, pour tout* $T > 0$, *chaque solution de (*2.12*) vérifiant* $B_2^* p(s) = 0$ *et* $D_1^* q(s) = 0$, $P - p.s.$, *pour tout* $s \in [0, T]$ *est réduite à* 0.

*En plus, l'équation (*2.11*) est approximativement 0-contrôlable si et seulement si, pour tout* $T > 0$, *chaque solution de (*2.12*) vérifiant* $B_2^* p(s) = 0$ *et* $D_1^* q(s) = 0$, $P - p.s.$, *pour tout* $s \in [0, T]$ *satisfait* $p(0) = 0$.

Une étape essentielle pour établir ce résultat est

Proposition 9 *Si* $X^{x,u}$ *est l'unique solution "mild" de (*3.15*) associée au processus de contrôle admissible* u *et si* (Y, Z) *est l'unique solution au sens "mild" du système dual (*3.16*), alors on a*

$$E\left[\langle X_T^{x,u}, Y_T \rangle\right] = E\left[\langle x, Y_0 \rangle\right] + E\left[\int_0^T \langle Bu_s, Y_s \rangle \, ds\right]. \tag{3.18}$$

Le résultat de dualité peut être récrit à l'aide du noyau de viabilité rétrograde introduit par Buckdahn, Quincampoix, Răşcanu[10]. Rappelons ici cette notion

Définition 6 *Soit* K *un sous-ensemble non-vide, convexe*[1]*, fermé de* H.

(i) Le processus stochastique $\{Y_t,\ t \in [0, T]\}$ *est viable dans* K *si et seulement si* $Y_t \in K$, *P-p.s., pour tout* $t \in [0, T]$.

(ii) L'ensemble K *a la propriété de viabilité stochastique rétrograde au temps* T *par rapport à (*3.16*) si pour toute condition finale* $\eta \in L^2(\Omega, \mathcal{F}_T, P; K)$, *la solution* $\{Y_t^\eta,\ t \in [0, T]\}$ *de (*3.16*) est viable dans* K.

[1] Rappelons que, d'après [10] un ensemble fermé viable pour l'équation (3.16) est nécessairement convexe. Ceci justifie de se restreindre aux convexes.

(iii) Le plus grand sous-ensemble convexe de K qui a la propriété de viabilité stochastique rétrograde au temps T est appelé noyau de viabilité stochastique rétrograde de K.

En utilisant cette notion on obtient

Proposition 10 *L'équation linéaire stochastique (*3.15*) est approximativement contrôlable si et seulement si, pour tout temps fini $T > 0$, le noyau de viabilité stochastique rétrograde de $Ker\ B^* = \{y \in H :\ B^*y = 0\}$ au temps T, par rapport à (*3.16*) est le sous-espace trivial* $\{0\}$.

Remarque 3 *Si W est un mouvement Brownien uni-dimensionnel, $B \in \mathcal{L}(H)$, et si C est un opérateur linéaire sur H (qui peut être non-borné) tels que $A^*B^* = B^*A^*$ et $B^*C^* = C^*B^*$, alors (*3.15*) a la propriété de contrôlabilité approchée si et seulement si l'espace image $\mathcal{R}(B)$ est dense dans H.*

3.2.3 Une condition nécessaire

On suppose, en plus de A.1 et A.2,

(A.3) *L'opérateur linéaire A est le générateur d'un semi-groupe exponentiellement stable d'opérateurs.*

Dans le cas du système stochastique (3.15), on montre la condition nécessaire suivante

Proposition 11 *Si (*3.15*) est approximativement contrôlable, alors pour tout* $y \in D(A^*)$ *tel que* $y \neq 0$*, et tout* $\alpha < 0$,

$$|B^* y| + |(A^* - \alpha I) y| > 0. \tag{N1}$$

Remarque 4 *Si on suppose que* A *est le générateur d'un* C_0*-semigroupe sur* H, W *est un mouvement Brownien 1-dimensionnel,* C *est un opérateur linéaire borné sur* H, $C \in \mathcal{L}(H)$*, l'espace de contrôle* U *est un sous-espace borné, fermé d'un espace de Hilbert réel et séparable* V*, et l'opérateur* $B \in \mathcal{L}(V; H)$ *, alors, on peut donner une autre condition nécessaire pour la contrôlabilité approchée de (*3.15*). Plus précisément, si (*3.15*) est approximativement contrôlable, alors*

$$|B^* y| + |(A^* + \lambda C^* - \alpha I) y| > 0, \tag{3.19}$$

pour tout $y \in D(A^*)$ *tel que* $y \neq 0$*, et tout* $(\lambda, \alpha) \in \mathbb{R} \times \mathbb{R}_-$.

Chapitre 4
Assurance, réassurance et paiement de dividendes

4.1 Présentation du problème

L'un des problèmes principaux des compagnies d'assurances est de trouver une stratégie permettant de satisfaire les demandes qui proviennent d'une part comme conséquence directe des contrats d'assurance conclus et, d'autre part, des actionnaires. Pour réduire le risque intrinsèque du travail d'assurance et se protéger de grandes pertes, les sociétés d'assurances utilisent la réassurance. Ce processus consiste à céder une partie des primes d'assurance vers une tierce partie ; en échange, la société de réassurance s'oblige a couvrir un certain niveau des pertes qui peuvent se produire. Il est évident que la société d'assurance détient le contrôle à la fois sur le niveau de réassurance et sur le niveau des dividendes payés aux actionnaires. Cela justifie le cadre du contrôle stochastique.

Les codes d'assurance spécifient qu'à chaque instant, les sociétés d'assurances doivent être capable de couvrir tout sinistre qui s'est produit ou qui peut être raisonnablement prévu. D'habitude, la marge de solvabilité est calculée par rapport aux primes d'assurances ainsi qu'aux pertes moyennes. Les formules qui doivent être prises en compte dépendent des spécificités liées au type d'assurance concernée. Ainsi, il existe plusieurs méthodes. Par exemple, le Code des Assurances français (R334-13) stipule que, dans le cas des assurances-vie, la marge de solvabilité (remplacée par Solvency Capital Requirement pour

Solvency II) doit être supérieure au résultat obtenu en multipliant 0,3% du capital sous risque avec le rapport entre le capital sous risque après cession en réassurance et ce capital avant cession calculé pour l'exercice précédent. Ce rapport ne peut pas être inférieur a 50%. Du point de vue mathématique, cela signifie qu'a chaque instant t, le résultat obtenu multipliant une constante ζ_0(donnée par l'expérience et le type d'assurance pratiquée) par la perte moyenne par contrat et par le nombre de contrats n_t ne peut pas être supérieur à la fortune de la société d'assurance.

$$\zeta_0 \times n_t \times \text{perte moyenne} \leq \text{fortune au temps } t. \tag{4.20}$$

L'évolution du capital de la société d'assurance sera modélisée par une équation stochastique par rapport à une mesure aléatoire μ (qui correspond à un processus de Poisson composé)

$$X_t^{x,u,L} = x + \int_0^t n_s p(u_s) ds - \int_0^{t+} \int_B f(n_s, y \wedge u_s) \mu(dsdy) - \int_0^t dL_s. \tag{4.21}$$

Dans l'équation ci-dessus u décrit un processus de contrôle de la réassurance, L le montant des dividendes, p est la prime par contrat, et f décrit les dédommagements payés en tenant compte du nombre des contrats et de la réassurance. Le but de cette partie est d'optimiser, par rapport aux processus admissibles de réassurance et paiement de dividendes, la valeur moyenne actualisée des dividendes payés aux actionnaires avant l'instant de ruine

$$V(x) = \sup_{u,L} E\left[\int_0^{\tau} e^{-rs} dL_s\right].$$

On montre que la fonction valeur ainsi définie est la solution de viscosité de l'inégalité variationnelle de Hamilton-Jacobi-Bellman associée. On montre également un résultat d'unicité de la solution de viscosité. Cette approche avait déjà été utilisée par Mnif, Sulem [37] mais dans un modèle de risque collectif où un seul contrat d'assurance est retenu. On montre par un simple exemple que, dans le cas où un seul contrat est admis indépendamment de la fortune de la société d'assurance, la fonction valeur est supérieure à une quantité strictement positive (indépendamment du capital initial arbitrairement petit). L'originalité de notre travail réside dans le fait de considérer plusieurs contrats d'assurances, et de tenir compte des réglementations en vigueur en matière d'assurance. Cela nous permet d'éviter le comportement déviant décrit ci-dessus pour la fonction valeur. De plus, la régularité de la fonction valeur obtenue dans ce cas nous permet de montrer, par un argument standard, le principe de programmation dynamique, tandis que cela était une hypothèse dans Mnif, Sulem [37].

4.2 Les résultats

4.2.1 Introduction

On introduit un espace de probabilité complet $(\Omega, \mathcal{F}, P)$. Les sinistres seront modélisées par un processus de Poisson composé donné par une mesure aléatoire $\mu(dtdy)$ sur $\mathbb{R}_+ \times B$, où $B \subset \mathbb{R}_+ \setminus \{0\}$. On suppose également que le compensateur de μ a la forme $dt\pi(dy)$, où $\pi(dy) = \beta G(dy)$ pour une mesure de probabilité $G\,(dy)$ et pour une constante positive β.

On introduit la variable aléatoire Y distribuée $G(dy)$.

La filtration $(\mathcal{F}_t)_{t\geq 0}$ sera la filtration générée par la mesure aléatoire μ. On appelle niveau de rétention, tout processus $(\mathcal{F}_t)$-adapté $(u_t)_{t\geq 0}$ qui spécifie que, étant donné un sinistre d'intensité y au temps $t \geq 0$, l'assureur direct doit couvrir $y \wedge u_t$ tandis que la société de réassurance couvrira l'excès de perte $(y - u_t)^+$.

Si la société choisit le niveau de rétention u_t, la prime par contrat est donnée comme pour Asmussen, Højgaard, Taksar [1],

$$p(u_t) = (1 + k_1)\beta\nu - (1 + k_2)\beta E\left[f(1, (Y - u_t)^+)\right] \text{ for all } t \geq 0, \tag{4.22}$$

où $0 \leq k_1 < k_2$ et

$$\nu = \int_B f(1, y)G(dy) = E[f(1, Y)]. \tag{4.23}$$

Si on pose

$$a = \frac{1}{\zeta_0 \nu},$$

alors, (4.20) implique l'évolution suivante pour le capital de la société d'assurance

$$X_t^{x,u,L} = x + a\int_0^t X_s^{x,u,L} p(u_s)ds - \int_0^{t+}\int_B f(aX_{s-}^{x,u,L}, y \wedge u_s)\mu(dsdy) - \int_0^t dL_s, \tag{4.24}$$

où L est le processus $\mathcal{F}_t-$adapté qui décrit les dividendes payés avant t. On considère la fonctionnelle de coût

$$J(x, u, L) = E\left[\int_0^\tau e^{-rs} dL_s\right], \tag{4.25}$$

où r est un facteur d'escompte et τ est le temps de ruine

$$\tau = \inf\left\{t \geq 0 : X_t^{x,u,L} \leq 0\right\}.$$

La fonction V sera définie comme le maximum sur les couples admissibles (u, L) de cette fonctionnelle J.

4.2.2 Hypothèses et résultats principaux

On travaille sous l'hypothèse suivante sur la fonction f qui décrit les dédommagements à payer en cas de sinistre

(A1) La fonction $f : \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}_+$ vérifie :

- $f(\cdot, y)$ est convexe, non-décroissante et $f(0, y) = 0$ pour tout $y \in \mathbb{R}_+$;

- $f(x, \cdot)$ est croissante et $f(x, 0) = 0$;

- $f(x, y) > 0$ si $x > 0$ et $y > 0$;

- f est uniformément continue sur $\mathbb{R}_+ \times \mathbb{R}$;

- $f(x, y)$ a la propriété de Lipschitz en x, uniformément en $y \in \mathbb{R}_+$.

(A2) Le facteur d'escompte r dans (4.25) satisfait

$$r > \frac{2(1 + k_1)\beta}{\zeta_0}$$

Définition 7 *On dit que le couple (u, L) est admissible si*

i) u, L sont $\mathcal{F}_t-$adapté,

ii) $u_t \geq \underline{u}$ où $p(\underline{u}) = \beta E[f(1, Y \wedge \underline{u})]$,

iii) L est càdlàg, non-décroissant, $L_{0-} = 0$ et $L_t - L_{t-} \leq X^{x,u,L}_{t-}$ pour presque tout (t, ω).

Remarque 5 *La condition ii) décrit le fait que les primes d'assurance doivent être suffisamment élevées pour couvrir le sinistre moyen, tandis que iii) permet a la société d'assurance à l'instant t de payer aux actionnaires au plus le capital dont elle dispose au temps t.*

Le premier pas pour obtenir la régularité de la fonction valeur est

Proposition 12 *Si $0 \leq x \leq x'$ sont deux capitaux initiaux et si (u, L) est admissible pour x, alors (u, L) est également admissible pour x'.*

On utilise cette proposition pour obtenir

Proposition 13 *La fonction valeur V est non-décroissante, a la propriété de Lipschitz et vérifie*

$$V(x) \leq Kx, \tag{4.26}$$

pour une constante assez grande K.

Le résultat principal est le suivant

Théorème 6 *La fonction valeur est l'unique solution de viscosité de croissance au plus linéaire de l'inégalité variationnelle de Hamilton-Jacobi-Bellman associée*

$$\begin{cases} \max\{H(x, V, V'(x)), 1 - V'(x)\} = 0 \text{ in } \mathbb{R}_+^*, \\ V(0) = 0, \end{cases} \tag{4.27}$$

où

$$\begin{aligned} & H(x, V, q) \qquad (4.28) \\ = \; & \sup_{u \geq \underline{u}} \left\{ -rV(x) + axp(u)q + \int_B \left[V(x - f(ax, y \wedge u)) - V(x)\right] \pi(dy) \right\}. \end{aligned}$$

On rappelle ici la notion de soussolution (respectivement sursolution, solution) de viscosité de l'inégalité variationnelle (4.27) (voir, par exemple [43]) :

Définition 8 *(i) Une fonction semi-continue inférieurement (respectivement supérieurement semi-continue) v est appelée sursolution de viscosité (soussolution) de (4.27) si $v(0) \geq 0$ (≤ 0) et*

$$\max\{H(x, \varphi, \varphi'(x)), 1 - \varphi'(x)\} \leq 0,$$

(respectivement ≥ 0*) pour tout* $\varphi \in C^{1,1}(\mathbb{R}_+)$ *telle que* $v - \varphi$ *admet un minimum (maximum) global en* $x > 0$.

(ii) Une fonction v *est solution de viscosité de (*4.27*) si elle est simultanément sursolution st soussolution de viscosité pour (*4.27*).*

Pour montrer le résultat principal, on utilise une approche standard. D'abord, la régularité de la fonction valeur V nous permet de prouver qu'elle vérifie le principe de programmation dynamique associée. En suite, on utilise des fonctions test $C^{1,1}$ et le principe de programmation dynamique pour montrer que V satisfait, au sens de viscosité, (4.27). Pour l'unicité, on montre un lemme de substitution ainsi qu'un résultat de comparaison entre les sur et soussolutions de viscosité de cette inéquation variationnelle.

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Non compact-valued stochastic control under state constraints

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Abstract

In the present paper, we study a necessary condition under which the solutions of a stochastic differential equation governed by unbounded control processes, remain in an arbitrarily small neighborhood of a given set of constraints. We prove that, in comparison to the classical constrained control problem with bounded control processes, a further assumption on the growth of control processes is needed in order to obtain a necessary and sufficient condition in terms of viscosity solution of the associated Hamilton-Jacobi-Bellman equation. A rather general example illustrates our main result.

Résumé

Dans cet article, nous étudions une condition nécessaire sous laquelle les solutions d'une équation différentielle stochastique régie par un processus de contrôle non-borné restent dans un voisinage arbitrairement petit d'un ensemble donné de contraintes. On montre que, par rapport au problème classique de contrôle sous contraintes avec des processus de contrôle bornés, afin d'obtenir une condition nécessaire et suffisante de la viabilité en termes de solution de viscosité de l'équation de Hamilton-Jacobi-Bellman associée, on a besoin d'une hypothèse supplémentaire sur la croissance du processus de contrôle. Un exemple assez général illustre notre résultat principal.

Key words: Stochastic control, Viscosity Solution, Stochastic Viability

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1 Introduction

Let be given a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions of completeness and right-continuity, and a d-dimensional $(\mathcal{F}_t)$-Brownian motion W on this space. We consider the following stochastic differential equation:

$$\begin{cases} dX^{x,v(\cdot)}(t) = b(X^{x,v(\cdot)}(t), v(t))dt + \sigma(X^{x,v(\cdot)}(t), v(t))dW(t), \ t \geq 0, \\ X^{x,v(\cdot)}(0) = x \in \mathbb{R}^n, \end{cases} \tag{1}$$

which is governed by a control process $v(\cdot)$ taking its values in an unbounded control state space.

Given a closed subset K of $\mathbb{R}^n$, the objective of our paper is to obtain necessary and sufficient conditions on the coefficients of equation (1) under which, for every starting point $x \in \mathbb{R}^n$, there exists an admissible control process $v(\cdot)$ that keeps the solution process $X^{x,v(\cdot)}$ inside the set K or, at least, in an arbitrarily small neighborhood of K. This property is called viability of K with respect to the control system (1) and has been extensively studied in [1], [3], [4], [5] and [7] for stochastic systems with compact-valued control processes admitting strong solutions and in [9] for the context of weak solutions. The methods used rely either on stochastic contingent cones (in [1], [12]), or on viscosity solutions (in [3], [4], [5]). We also recall related works on invariance property of K [13] for a study via contingent cones and [9] for an approach through the generalization of Doss-Sussman transformation.

The viscosity approach expresses the viability property by a criterion involving the value function

$$V(x) = \inf E\left[\int_0^\infty e^{-Cs} d_K^2(X^{x,v(\cdot)}(s))ds\right], \ x \in \mathbb{R}^n, \tag{2}$$

where d_K is the distance function to the closed set $K \subset \mathbb{R}^n$ and the infimum is taken over the family of all admissible control processes defined over $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$. In order to guarantee the existence of an optimal control in a weak sense for (2), the authors of [3] supposed that the set $\left\{\left(\frac{1}{2}\sigma\sigma'(x,u), b(x,u)\right), \ u \in U\right\}$ is convex and compact for all $x \in \mathbb{R}^n$. In the case of viability it is this optimal control which keeps the process $X^{x,v(\cdot)}$ inside of K. If one has only ε-optimal controls which are keeping $X^{x,v(\cdot)}$ in an arbitrary small neighborhood of K, one is speaking about the so-called ε-viability. In both cases, in the framework of a compact control state space, it has been shown in the papers cited above, that the viability property of K with respect to the control system (1) is equivalent to the fact that d_K^2 is a viscosity supersolution of the Hamilton-Jacobi-Bellman equation associated to the control problem (2).

In the case of an unbounded control state space, without any assumption of L^p-boundedness of the control process, things look quite different. In our paper, we discuss an example with $K = \{-1, 0\} \subset \mathbb{R}$ where, although the value function V is null on K, we are far from a situation which could be called ε-viability. Indeed, we show that for every $\varepsilon > 0$ there exists an ε-optimal control process $v^\varepsilon(\cdot)$ for (2) which pushes $X^{-1,v^\varepsilon(\cdot)}$ from -1 to the point 0 of K. Recall that the interval $(-1, 0)$ does not belong to K. This example shows that some supplementary assumption is needed in order to guarantee ε-viability for the case that d_K^2 is a viscosity supersolution. Observing that the above mentioned family of control processes $\{v^\varepsilon(\cdot), \varepsilon > 0\}$ is only bounded in L^1 but not in L^p, for $p > 1$, we introduce an additional equation to the control system (1). This equation acts as a regulatory term and can be interpreted as supplementary running cost related to the control. It has the consequence that we can construct families $\{v^\varepsilon(\cdot), \varepsilon > 0\}$ of ε-optimal controls that are bounded in L^p (for some $p > 1$). To be more precise, we study the following stochastic control system:

$$\begin{cases} dX^{x,v(\cdot)}(t) = b(X^{x,v(\cdot)}(t), v(t))dt + \sigma(X^{x,v(\cdot)}(t), v(t))dW(t); \\ dY^{x,y,v(\cdot)}(t) = f(X^{x,v(\cdot)}(t), Y^{x,y,v(\cdot)}(t), v(t))dt, \ t \geq 0; \\ X^{x,v(\cdot)}(0) = x \in \mathbb{R}^n; \ Y^{x,y,v(\cdot)}(0) = y \in \mathbb{R}, \end{cases} \tag{3}$$

with $f(x, y, v) \geq |v|^p - \beta(x)$, where β is a continuous function. We characterize the ε-viability for this system with the help of the associated Hamilton-Jacobi-Bellman equation. Our results will be illustrated by an example of an unbounded cylinder K. The necessary and sufficient condition for the ε-viability of K looks similar to that of the compact-valued control case. But in this example, it turns out that, by a suitable choice of the coefficients, the feedback control process is necessarily unbounded. We obtain, therefore, an example where the classical methods using the compactness of the control state space fail to hold, while it is covered by our approach.

We now focus on the structure of the paper. The first section illustrates, by means of a deterministic control system, why, in general, the $\varepsilon-$viability property does not imply a viscosity-type condition in the case of unbounded control processes. The second section is devoted to the study of the ε-viability property of the system (3) for unbounded control processes. A sufficient and necessary criterion of ε-viability is given in terms of the associated Hamilton-Jacobi-Bellman equation. The paper is closed by the study of explicit conditions for the ε-viability in the case of a cylinder.

2 Existence of stochastic control under state constraints

2.1 Preliminaries

Let $\nu = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P, W)$ be a reference probability system consisting of a complete probability space $(\Omega, \mathcal{F}, P)$, a filtration satisfying the usual assumptions of completeness and right-continuity, and a d-dimensional $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion W defined on this space. We consider the following stochastic differential equation:

$$\begin{cases} dX^{x,v(\cdot)}(t) = b(X^{x,v(\cdot)}(t), v(t))dt + \sigma(X^{x,v(\cdot)}(t), v(t))dW(t),\ t \geq 0, \\ X^{x,v(\cdot)}(0) = x \in \mathbb{R}^n, \end{cases} \tag{4}$$

where $v(\cdot)$ is a control processes taking its values in a metric space U.

We recall the definitions of viability, respectively ε-viability:

Definition 1 *A closed set $K \subset \mathbb{R}^n$ enjoys the viability property with respect to (4) if, for all $x \in K$, there exist a probability space $(\Omega, \mathcal{F}, P)$, a d-dimensional $(\mathcal{F}_t)_{t\geq 0}-$Brownian motion W defined on this space and a U-valued $(\mathcal{F}_t)$-progressively measurable control processes $u(\cdot)$, such that, $P-a.s.$, $X^{x,u(\cdot)}(t) \in K$, for all $t \geq 0$.*

Moreover, we say that K is $\varepsilon-$viable with respect to (4) if, for all $\varepsilon > 0$, $x \in K$, there exist a probability space $(\Omega, \mathcal{F}, P)$, a d-dimensional $(\mathcal{F}_t)_{t\geq 0}-$Brownian motion W defined on this space and a U-valued $(\mathcal{F}_t)$-progressively measurable control processes $u^\varepsilon(\cdot)$, such that, $P-a.s.$, $d_K(X^{x,u^\varepsilon(\cdot)}(t)) \leq \varepsilon$, for all $t \geq 0$.

In the case where U is compact (see [3]), in order to characterize the ε-viability property for a closed set $K \subset \mathbb{R}^n$, the following value function has been introduced:

$$V(x) = \inf_{v\in\mathcal{A}} E\left[\int_0^\infty e^{-Cs} d_K^2(X^{x,v(\cdot)}(s))ds\right]. \tag{5}$$

where $C = (L+1)^2$ (L beeing the Lipschitz constant for the coefficients). The infimum in (5) is taken over the family $\mathcal{A}$ of U-valued $(\mathcal{F}_t)$-progressively measurable control processes. We also introduce the associated Hamilton-Jacobi-Bellman equation

$$\begin{aligned} 0 = \sup_{v\in U} &\left\{ -\langle D_x u(x), b(x,v)\rangle - \frac{1}{2}Tr[D_{xx}^2 u(x)\sigma\sigma^*(x,v)]\right\} \\ &+Cu(x) - d_K^2(x),\ x \in \mathbb{R}^n, \end{aligned} \tag{6}$$

under the following assumptions on the coefficient functions $b : \mathbb{R}^n \times U \longrightarrow \mathbb{R}^n$

and $\sigma : \mathbb{R}^n \times U \longrightarrow \mathbb{R}^{n\times d}$:

(i) The functions b, σ are of at most linear growth,

(ii) The coefficients b, σ are uniformly continuous on $\mathbb{R}^n \times U$ and Lipschitz in $x \in \mathbb{R}^n$, uniformly in $u \in U$.

The result in [3] may be resumed as follows:

Proposition 2 *([3] Theorem 2) Under (i) and (ii), the following conditions are equivalent:*

(i) For all $x \in K$, $V(x) = 0$;

(ii) The function $d_K^2(\cdot)$ is a viscosity supersolution for (6).

2.2 Viability for unbounded control: a counter example.

Let us now concentrate on the case $U = \mathbb{R}^k$ and see by means of a simple example which the difficulties encountered are when trying to obtain an analogous assertion:

Example 3 *We consider $d = n = k = 1$, $\sigma : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, with $\sigma(x, v) = 0$, $x, v \in \mathbb{R}$, and let $b : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, with $b(x, v) = |x| + |v|$, $x, v \in \mathbb{R}$. As space of admissible controls we choose $\mathcal{A} = L^2_{loc}(\mathbb{R}, dt)$.*

We deal with the following deterministic control system

$$\begin{cases} dX^{x,v(\cdot)}(t) = (|X^{x,v(\cdot)}(t)| + |v(t)|)dt, \\ \qquad\qquad X^{x,v(\cdot)}(0) = x, \end{cases} \tag{7}$$

$x \in \mathbb{R}$, $v(\cdot) \in \mathcal{A}$. We consider the set $K = \{-1, 0\} \subset \mathbb{R}$ and we define the following value function

$$V(x) = \inf_{u\in\mathcal{A}} \int_0^\infty e^{-Cs} d_K^2(X^{x,u(\cdot)}(s))ds.$$

Then, the associated Hamilton-Jacobi-Bellman equation (6) takes the following form:

$$\inf_{v\in\mathbb{R}} \{DV(x)(|x| + |v|)\} + d_K^2(x) - CV(x) = 0. \tag{8}$$

It can be easily seen that $V(0) = 0$. Furthermore, we shall prove that $V(-1) = 0$. For this end, for any $v > 0$, let us define the following admissible control:

$$u^v(t) = \begin{cases} v, \text{ if } t \in [0, \ln(\frac{1+v}{v})); \\ 0, \text{ if } t \geq \ln(\frac{1+v}{v}). \end{cases}$$

Then, the solution associated with the control $u^v(\cdot)$ and starting at -1 has the following evolution:

$$X^{-1,u^v(\cdot)}(t) = \begin{cases} v - (1+v)e^{-t}, \text{ if } t \in [0, \ln(\frac{1+v}{v})), \\ 0 \qquad\qquad\qquad , \text{ if } t \geq \ln(\frac{1+v}{v}). \end{cases}$$

Taking into account that for $t \in [0, \ln(\frac{1+v}{v}))$, $X^{-1,u^v(\cdot)}(t) \in [-1, 0)$, we deduce that

$$d_K(X^{-1,u^v(\cdot)}(t)) \leq \frac{1}{2}, \quad t \in [0, \ln(\frac{1+v}{v})).$$

On the other hand, for $t \geq \ln(\frac{1+v}{v})$, $X^{-1,u^v(\cdot)}(t) = 0 \in K$. Consequently, for all $v > 0$,

$$V(-1) \leq \int_0^\infty e^{-Ct} d_K^2(X^{-1,u^v(\cdot)}(t))dt \leq \frac{1}{4}\ln(\frac{1+v}{v}).$$

We let now $v \to +\infty$ to get $V(-1) = 0$.

Although $V = 0$ on K, d_K^2 is not a viscosity supersolution for (8). To see this, it suffices to take the test function $\varphi(y) = (y+1)^2$, $y \in \mathbb{R}$ and to compare it to V. Indeed, φ is of class C^2 and in spite of the fact that $d_K^2(y) - \varphi(y) = (y+1)^2 \wedge y^2 - (y+1)^2$ admits a local minimum at $y_0 = -1 + \frac{1}{C+1}$ (recall that $C > 1$) we have

$$-\inf_{v\in\mathbb{R}}(D\varphi(y_0)(|y_0| + |v|)) - d_K^2(y_0) + C\varphi(y_0) = -\frac{1}{C+1} < 0.$$

This example shows that we should not expect the Proposition 2 to hold true in the general case of an unbounded control. On the other hand, recall that, for any $v > 0$, we have defined

$$u^v(t) = \begin{cases} v, \text{ if } t \in [0, \ln(\frac{1+v}{v})); \\ 0, \text{ if } t \geq \ln(\frac{1+v}{v}). \end{cases}$$

Therefore, we may infer

$$\int_0^\infty e^{-Cs}|u^v(s)|ds = \frac{v}{C}\frac{(1+v)^C - v^C}{(1+v)^C}.$$

Let us emphasize that the family of controls $\{u^v : v > 0\}$ is bounded in $L^1(\mathbb{R}, e^{-Cs}ds)$. Furthermore, for any $p > 1$, the family of controls $\{u^v : v > 0\}$ is no longer bounded in $L^p(\mathbb{R}, e^{-Cs}ds)$. Therefore it is straightforward to consider an additional equation allowing to obtain a condition to be (locally) assimilated to L^p bound, where $p > 1$ is chosen great enough (in our proof, for technical arguments, we need p such that we may find $a > 2$ satisfying $p > \max\left\{3a, \frac{a+4}{a-2}\right\}$; we may choose, for example, $p > \frac{17}{2}$). Thus, it seems to be natural to impose further conditions on the coefficients as well as on the control to get an assertion analogous to the result for the compact-valued control processes. This will be done in the next section.

3 $\mathbf{L}^p$ stochastic controls

Following the remarks in the previous example, from now on, we consider the following extended stochastic control system:

$$\begin{cases} dX^{x,v(\cdot)}(t) = b(X^{x,v(\cdot)}(t), v(t))dt + \sigma(X^{x,v(\cdot)}(t), v(t))dW(t), \\ dY^{x,y,v(\cdot)}(t) = f(X^{x,v(\cdot)}(t), Y^{x,y,v(\cdot)}(t), v(t))dt, \ t \geq 0, \\ X^{x,v(\cdot)}(0) = x \in \mathbb{R}^n; \ Y^{x,y,v(\cdot)}(0) = y \in \mathbb{R}, \end{cases} \tag{9}$$

where we make the following standard assumptions on the coefficients:

(A.1) *There exist a constant $L > 0$ such that the coefficients $b : \mathbb{R}^n \times \mathbb{R}^k \longrightarrow \mathbb{R}^n$, $\sigma : \mathbb{R}^n \times \mathbb{R}^k \longrightarrow \mathbb{R}^{n\times d}$, $f : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^k \longrightarrow \mathbb{R}$ satisfy*

$$\begin{aligned} |b(x,u) - b(x',u)| &\leq L|x - x'| \\ |\sigma(x,u) - \sigma(x',u)| &\leq L|x - x'| \\ |f(x,y,u) - f(x',y',u)| &\leq L(|y - y'| + |x - x'|) \end{aligned}$$

for any $x, x' \in \mathbb{R}^n$, $y, y' \in \mathbb{R}$, $u \in \mathbb{R}^k$.

(A.2) *b and σ uniformly continuous on $\mathbb{R}^n \times \mathbb{R}^k$ and they have at most linear growth, i.e. there exists some $L > 0$ such that*

$$\begin{aligned} |b(x,u)| &\leq L(1 + |x| + |u|) \\ |\sigma(x,u)| &\leq L(1 + |x| + |u|) \end{aligned}$$

for any $x \in \mathbb{R}^n$, $u \in \mathbb{R}^k$.

The function f is uniformly continuous, $\sup_{x\in\mathbb{R}^n, y\in\mathbb{R}} |f(x,y,0)| \leq L$, *and, for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^k$, $f(x,\cdot,u)$ is nonnegative on $[l,\infty)$. Moreover, for* *some continuous function* $\beta : \mathbb{R}^n \to \mathbb{R}$ and $p > 2$ great enough,

$$f(x, y, u) \geq |u|^p - \beta(x),$$

for all $x \in \mathbb{R}^n$, $y \in \mathbb{R}$, $u \in \mathbb{R}^k$.

Let $\mathcal{A}$ denote the class of all admissible control processes, that is the class of the $\mathbb{R}^k-$valued progressively measurable processes $v(\cdot)$ satisfying

$$E\left[\int_0^T |v(s)|^p ds\right] < \infty, \text{ for all } T > 0 .$$

In fact, this integrability condition is needed in order to give a meaning to our control system (9). Let K be an arbitrary closed subset of $\mathbb{R}^n$. We denote by B_l the interval $[-l, l] = \{y \in \mathbb{R}, |y| \leq l\}$, where $l > 0$ is fixed.

Furthermore, we assume that

(A.3)

$$f(x, y, v) = f(\pi_K(x), y, v),$$

for some measurable selection $\pi_K : \mathbb{R}^n \longrightarrow K$, *such that* $\pi_K(x) \in \Pi_K(x) = \{y \in K : d_K(x) = |x - y|\}$ *for all* $x \in \mathbb{R}^n$, $y \geq l$, $u \in \mathbb{R}^k$.

Remark 4 *If* $f(x, y, v) = f(y, v)$, *the previous condition is obviously satisfied.*

For any admissible control process $v(\cdot)$ and any pair $(x, y) \in \mathbb{R}^n \times \mathbb{R}$ we introduce the associated cost functional

$$J(x, y; v(\cdot)) = E\left[\int_0^\infty e^{-Cs}\left(d_K^2(X^{x,v(\cdot)}(s)) \wedge 1 + d_{B_l}(Y^{x,y,v(\cdot)}(s))\right) ds\right] \quad (10)$$

for some large enough $C > 0$, and we wish to minimize the cost functional over the family of admissible controls:

$$V(x, y) = \inf_{v \in \mathcal{A}} J(x, y; v(\cdot)). \quad (11)$$

Essential properties of the value function V are given by the following

Proposition 5 *The function* V *is real-valued and enjoys the Lipschitz property.*

PROOF. The fact that the function V does not take the value infinity follows easily from the observations that

$$V(x, y) \leq J(x, y; 0), \ (x, y) \in \mathbb{R}^n \times \mathbb{R},$$

and

$$J(x,y;0) \le C_L\left(d_{B_l}(y)+1\right), \ (x,y)\in\mathbb{R}^n\times\mathbb{R},$$

where the latter inequality is obtained by standard estimates for solutions of (9).

To prove the second assertion we remark that, for (x,y), $(x',y')\in\mathbb{R}^n\times\mathbb{R}$,

$$\begin{aligned}
&|J(x',y';v(\cdot))-J(x,y;v(\cdot))| \\
\le&\frac{2}{\sqrt{C}}\left(E\left[\int_0^\infty e^{-Cs}|X^{x',v(\cdot)}(s)-X^{x,v(\cdot)}(s)|^2ds\right]\right)^{\frac{1}{2}} \\
&+E\left[\int_0^\infty e^{-Cs}|Y^{x,y,v(\cdot)}(s)-Y^{x',y',v(\cdot)}(s)|ds\right].
\end{aligned}$$

Thus, applying the Itô formula and then the Gronwall inequality, we obtain

$$|J(x',y';v(\cdot))-J(x,y;v(\cdot))|\le C_L(|x'-x|+|y'-y|).$$

Here $C_L>0$ denotes a constant depending only on L. This leads to the property of V.$\square$

Let us now consider the following Hamilton-Jacobi-Bellman equation

$$\begin{aligned}
0=&\sup_{v\in\mathbb{R}^k}\{-<D_xV(x,y),b(x,v)>-D_yV(x,y)f(x,y,v) \\
&-\frac{1}{2}Tr[D^2_{xx}V(x,y)\sigma\sigma^*(x,v)]\}+CV(x,y)-d^2_K(x)\wedge 1-d_{B_l}(y)\ .
\end{aligned}\tag{12}$$

The reader is referred to [11] for further literature on the origin of this type of equation as well as its connection with the stochastic control system (9). Furthermore, it is well-known that V is the unique viscosity solution for the Hamilton-Jacobi-Bellman equation in the class of continuous functions with polynomial growth (cf [11]).

We introduce the function $F:\mathbb{R}^n\times\mathbb{R}\longrightarrow\mathbb{R}$ defined by $F(x,y)=d^2_K(x)\wedge 1+d_{B_l}(y)$.

We state now the main result of our paper.

Theorem 6 *Under (A.1)-(A.3), the following assertions are equivalent:*

(i) The set $K\times B_l$ enjoys the ε-viability property for (9);

(ii) The function $F:\mathbb{R}^n\times\mathbb{R}\longrightarrow\mathbb{R}$, $F(x,y)=d^2_K(x)\wedge 1+d_{B_l}(y)$, $(x,y)\in\mathbb{R}^n\times\mathbb{R}$, is a viscosity supersolution for (12).

In order to prove this Theorem, we need a property of the control processes in (11) which is discussed in the following Remark.

Remark 7 *Let $x \in \mathbb{R}^n$, $y \in \mathbb{R}$ and $v(\cdot) \in \mathcal{A}$ be such that $J(x,y;v(\cdot)) \leq V(x,y)+1$. Then, from (10) and (A2) it follows that*

$$E\left[\int_0^\infty e^{-Cs}|v(s)|^p ds\right] - \left(E\left[\int_0^\infty e^{-Cs}|v(s)|^p ds\right]\right)^{\frac{1}{p}} \leq C(V(x,y)+1) + l - y + \frac{\beta(x)}{C} + 1 + |x| \tag{13}$$

(where the constant C is chosen great enough). Indeed, for any control process $v(\cdot)$ such that $J(x,y;v(\cdot)) \leq V(x,y)+1$, we have

$$\begin{aligned} &-LE\left[\int_0^\infty e^{-Cs}|X^{x,v(\cdot)}(s)-x|ds\right] + E\left[\int_0^\infty e^{-Cs}|v(s)|^p ds\right] - \frac{\beta(x)}{C} \\ &\leq -LE\left[\int_0^\infty e^{-Cs}|X^{x,v(\cdot)}(s)-x|ds\right] + E\left[\int_0^\infty e^{-Cs}f(x,Y^{x,y,v(\cdot)}(s),v(s))ds\right] \\ &\leq E\left[\int_0^\infty e^{-Cs}f(X^{x,v(\cdot)}(s),Y^{x,y,v(\cdot)}(s),v(s))ds\right] \\ &\leq C(V(x,y)+1)+l-y. \end{aligned} \tag{14}$$

(Recall that L is the Lipschitz constant introduced in (A1)). We may apply standard arguments to estimate

$$\begin{aligned} E\left[\int_0^\infty e^{-Cs}|X^{x,v(\cdot)}(s)-x|ds\right] &\leq \frac{1}{\sqrt{C}}\sqrt{E\left[\int_0^\infty e^{-Cs}|X^{x,v(\cdot)}(s)-x|^2 ds\right]} \\ &\leq \frac{1}{\sqrt{C}}\left((1+|x|)^2 + \frac{1}{C}E\left[\int_0^\infty e^{-C_L s}|v(s)|^2 ds\right]\right)^{\frac{1}{2}} \end{aligned} \tag{15}$$

and, from Holder's inequality,

$$E\left[\int_0^\infty e^{-C_L s}|v(s)|^2 ds\right] \leq C'\left(E\left[\int_0^\infty e^{-Cs}|v(s)|^p ds\right]\right)^{\frac{2}{p}}$$

(C_L and C' stand for real constants that only depend on L and C). Substituting the latter inequality in (15), we obtain

$$E\left[\int_0^\infty e^{-Cs}|X^{x,v(\cdot)}(s)-x|ds\right] \leq \frac{1}{\sqrt{C}}\left(1+|x|+\left(E\left[\int_0^\infty e^{-Cs}|v(s)|^p ds\right]\right)^{\frac{1}{p}}\right). \tag{16}$$

Finally, we substitute (16) in (14) to get the announced inequality (13).

Consequently, $V(x,y)$ may be regarded as the infimum over the family $\mathcal{A}^1(x,y)$

of controls from $\mathcal{A}$ satisfying (13). Furthermore, it follows that if $r < p$ then

$$\lim_{n\to\infty} \sup_{v\in\mathcal{A}^1(x,y)} E\left[\int_0^\infty e^{-Cs}|v(s)|^r 1_{\{|v(s)|>n\}}\right] = 0,$$

i.e. $\mathcal{A}^1(x,y)$ is uniformly r-integrable, for all $r < p$.

We now come to the proof of Theorem 6:

PROOF. We first show that (i) implies (ii).

For this we consider arbitrary $(x,y) \in \mathbb{R}^n \times \mathbb{R}$ and $\varphi \in C^2(\mathbb{R}^n \times \mathbb{R})$ such that

$$F - \varphi \geq F(x,y) - \varphi(x,y) = 0, \tag{17}$$

and we claim that

$$\begin{aligned}0 \leq \sup_{v\in\mathbb{R}^k}\{&- < D_x\varphi(x,y), b(x,v) > -D_y\varphi(x,y)f(x,y,v) \\ &-\frac{1}{2}Tr[D^2_{xx}\varphi(x,y)\sigma\sigma^*(x,v)]\} + CF(x,y) - d^2_K(x)\wedge 1 - d_{B_l}(y)\ .\end{aligned}$$

The proof of this claim will be divided in three cases, given by $D_y\varphi(x,y) < 0$, $D_y\varphi(x,y) = 0$ and $D_y\varphi(x,y) > 0$.

(a) We first suppose $D_y\varphi(x,y) < 0$. The hypothesis of linear growth in v for the coefficients, assumption (A2) on f and the fact that $p > 2$ allow to conclude that

$$\begin{aligned}+\infty = \sup_{v\in\mathbb{R}^k}\{&- < D_x\varphi(x,y), b(x,v) > -D_y\varphi(x,y)f(x,y,v) \\ &-\frac{1}{2}Tr[D^2_{xx}\varphi(x,y)\sigma\sigma^*(x,v)]\} + CF(x,y) - d^2_K(x)\wedge 1 - d_{B_l}(y).\end{aligned}$$

From this, our claim follows easily.

(b) Next we suppose $D_y\varphi(x,y) = 0$. This can only happen if $y \in B_l$. For any real-valued function $g : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}$, we denote by $g_y : \mathbb{R}^n \longrightarrow \mathbb{R}$ the function given by

$$g_y(x') = g(x', y)$$

for any $x' \in \mathbb{R}^n$. With this notation we have

$$F_y - \varphi_y \geq F_y(x) - \varphi_y(x) = 0 \text{ on } \mathbb{R}^n \times \mathbb{R}, \tag{18}$$

$D_x\varphi_y(x) = D_x\varphi(x,y)$ and $D^2_{xx}\varphi_y(x) = D^2_{xx}\varphi(x,y)$. Set $\tilde{x} = \pi_K(x)$. Since $y \in B_l$, y coincides with its projection onto B_l.

For any arbitrarily given $\varepsilon > 0$ let $v^\varepsilon \in \mathcal{A}$ be such that

$$J(\widetilde{x}, y; v^\varepsilon(\cdot)) \leq \varepsilon^5.$$

For the sake of simplicity we introduce the following abbreviated notations

$$X^\varepsilon(t) = X^{x,v^\varepsilon(\cdot)}(t),\ \widetilde{X}^\varepsilon(t) = X^{\widetilde{x},v^\varepsilon(\cdot)}(t),\ t \geq 0,$$

and we put

$$\begin{aligned}
\tau^\varepsilon &= \inf\{t \geq 0 : |X^\varepsilon(t) - x| > 1\ \} \wedge \inf\{t \geq 0 : |\widetilde{X}^\varepsilon(t) - \widetilde{x}| > 1\ \}, \\
\tau^\varepsilon_s &= s \wedge \tau^\varepsilon,\ 0 \leq s \leq 1.
\end{aligned}$$

Then, we get from (18)

$$F_y(X^\varepsilon(\tau^\varepsilon_s)) - F_y(x) \geq \varphi_y(X^\varepsilon(\tau^\varepsilon_s)) - \varphi_y(x).$$

Hence,

$$\int_0^\varepsilon E\left[F_y(X^\varepsilon(\tau^\varepsilon_s)) - F_y(x)\right] ds \geq \int_0^\varepsilon E[\varphi_y(X^\varepsilon(\tau^\varepsilon_s)) - \varphi_y(x)]ds. \tag{19}$$

The left-hand term in (19) can be estimated as follows:

$$\begin{aligned}
&\int_0^\varepsilon E\left[F_y(X^\varepsilon(\tau^\varepsilon_s)) - F_y(x)\right] ds \\
\leq &\int_0^\varepsilon E\left[|X^\varepsilon(\tau^\varepsilon_s) - \widetilde{X}^\varepsilon(\tau^\varepsilon_s)|^2 \wedge 1\right] ds - (d_K^2(x) \wedge 1)\varepsilon \\
&+ C_x \int_0^\varepsilon E\left[d_K(\widetilde{X}^\varepsilon(\tau^\varepsilon_s)) \wedge 1\right] ds \\
\leq &\, C_x \int_0^\varepsilon E[d_K(\widetilde{X}^\varepsilon(s) \wedge 1]ds + C_x \int_0^\varepsilon E[|\widetilde{X}^\varepsilon(s) - \widetilde{X}^\varepsilon(\tau^\varepsilon_s)|1_{\{\tau^\varepsilon < \varepsilon\}}]ds \\
&\int_0^\varepsilon E\left[|X^\varepsilon(\tau^\varepsilon_s) - \widetilde{X}^\varepsilon(\tau^\varepsilon_s)|^2 \wedge 1\right] ds - (d_K^2(x) \wedge 1)\varepsilon,
\end{aligned} \tag{20}$$

where C_x is a constant depending only on x. We denote the four terms on the right hand of the above inequality with I_1, I_2, I_3, respectively I_4 and give appropriate estimates for each one. The choice of the admissible control $v^\varepsilon(\cdot)$ allows us to write, for all $\varepsilon < 1$,

$$\int_0^\varepsilon E[d_K(\widetilde{X}^\varepsilon(s)) \wedge 1]ds \leq e^C \int_0^\infty e^{-Cs} E[d_K(\widetilde{X}^\varepsilon(s)) \wedge 1]ds \leq e^C \frac{1}{\sqrt{C}} \varepsilon^{\frac{5}{2}}. \tag{21}$$

In order to evaluate the term I_2 in (20), we give two simple but useful Lemmata:

Lemma 8 *With the previous notations we have, for all $\varepsilon > 0$ small enough and all $1 < a < p$,*

$$P(\tau^\varepsilon < \varepsilon) \le A\varepsilon^{\frac{ap-a}{2p}},$$

where A is a constant independent of ε.

Lemma 9 *Let $p > q \ge 2$. Under the above assumptions, for any $s > 0$ small enough, we have*

$$E\left[|X^\varepsilon(s) - x|^q\right] \le C_q s^{\frac{p-q}{p}},$$

and

$$E\left[|\widetilde{X}^\varepsilon(s) - \widetilde{x}|^q\right] \le C_q s^{\frac{p-q}{p}},$$

where C_q is constant independent of s.

Let us now continue the proof of Theorem 6. The proof of the two Lemmata will be given afterward.

At this point we shall consider the term I_2 in (20). A simple application of Holder's inequality provides

$$\int_0^\varepsilon E[|\widetilde{X}^\varepsilon(s) - \widetilde{X}^\varepsilon(\tau_s^\varepsilon)|1_{\tau^\varepsilon<\varepsilon}]ds \le \left(\int_0^\varepsilon E[|\widetilde{X}^\varepsilon(s) - \widetilde{X}^\varepsilon(\tau_s^\varepsilon)|^2]ds\right)^{\frac{1}{2}} \sqrt{\varepsilon P(\tau^\varepsilon < \varepsilon)}.$$

Further, for $t \le 1$ we apply Lemma 9 to obtain

$$\begin{aligned}\int_0^\varepsilon E[|\widetilde{X}^\varepsilon(s) - \widetilde{X}^\varepsilon(\tau_s^\varepsilon)|^2]ds &\le 2\int_0^\varepsilon \left(E[|\widetilde{X}^\varepsilon(s) - \widetilde{x}|^2] + E[|\widetilde{X}^\varepsilon(\tau_s^\varepsilon) - \widetilde{x}|^2]\right) ds \\ &\le C_2\varepsilon^{\frac{2p-2}{p}}.\end{aligned}$$

This combined with Lemma 8 permits writing

$$\int_0^\varepsilon E[|\widetilde{X}^\varepsilon(s) - \widetilde{X}^\varepsilon(\tau_s^\varepsilon)|1_{\tau^\varepsilon<\varepsilon}]ds \le C_a\varepsilon^{\frac{(6+a)p-a-4}{4p}}. \tag{22}$$

In order to estimate the term I_3 in (20), we use standard methods based on Itô's formula and Gronwall's Lemma. This yields

$$\int_0^\varepsilon E\left[|X^\varepsilon(\tau_s^\varepsilon) - \widetilde{X}^\varepsilon(\tau_s^\varepsilon)|^2 \wedge 1\right] ds \le (d_K^2(x) \wedge 1)e^{(L^2+2L)\varepsilon}\varepsilon. \tag{23}$$

Then, substituting (21), (22) and (23) in (20) gives

$$\int_0^\varepsilon E\left[F_y(X^\varepsilon(\tau_s^\varepsilon)) - F_y(x)\right] ds \tag{24}$$
$$\le C_x e^C \frac{1}{\sqrt{C}} \varepsilon^{\frac{5}{2}} + C_x \varepsilon^{\frac{(6+a)p-a-4}{4p}} + (d_K^2(x) \wedge 1)\varepsilon \left(e^{(L^2+2L)\varepsilon} - 1\right).$$

To conclude the proof of **(b)** we have to provide an adequate estimate for $\int_0^\varepsilon E[\varphi_y(X^\varepsilon(\tau_s^\varepsilon)) - \varphi_y(x)]ds$. For simplicity, for $x' \in \mathbb{R}^n$ and $u \in C^2$, we write

$$\mathcal{L}_{(x',v)} u(x') = < D_x u(x'), b(x',v) > + \frac{1}{2} Tr\left[D_{xx}^2 u(x') \sigma\sigma^*(x',v)\right].$$

Using Itô's formula for $\varphi_y(X^\varepsilon(\tau_s^\varepsilon))$ we have

$$\begin{aligned}
&\int_0^\varepsilon E[\varphi_y(X^\varepsilon(\tau_s^\varepsilon)) - \varphi_y(x)]ds \\
&= \int_0^\varepsilon E\left[\int_0^{\tau_s^\varepsilon} \mathcal{L}_{(X^\varepsilon(r), v^\varepsilon(r))} \varphi_y(X^\varepsilon(r)) dr\right] ds \\
&= \int_0^\varepsilon E\left[\int_0^{\tau_s^\varepsilon} \mathcal{L}_{(x, v^\varepsilon(r))} \varphi_y(x) dr\right] ds - \theta(\varepsilon),
\end{aligned}$$

where the rest $\theta(\varepsilon)$ is defined by the last equality. On the other hand, the inequalities

$$\begin{aligned}
&| < D_x\varphi(X^\varepsilon(r), y), b(X^\varepsilon(r), v^\varepsilon(r)) > - < D_x\varphi(x,y), b(x, v^\varepsilon(r)) > | \\
&\le L|D_x\varphi(X^\varepsilon(r), y)||X^\varepsilon(r) - x| + \\
&\quad + |D_x\varphi(X^\varepsilon(r), y) - D_x\varphi(x,y)|(L + L|x| + L|v^\varepsilon(r)|).
\end{aligned} \tag{25}$$

and

$$\begin{aligned}
&\left|\frac{1}{2} Tr\left[D_{xx}^2\varphi(X^\varepsilon(r), y)\sigma\sigma^*(X^\varepsilon(r), v^\varepsilon(r)) - D_{xx}^2\varphi(x,y)\sigma\sigma^*(x, v^\varepsilon(r))\right]\right| \\
&\le C_L|D_{xx}^2\varphi(X^\varepsilon(r), y)|(1 + |x| + |X^\varepsilon(r) - x| + |v^\varepsilon(r)|)|X^\varepsilon(r) - x| \qquad (26) \\
&\quad + C_L|D_{xx}^2\varphi(X^\varepsilon(r), y)|(1 + |x| + |v^\varepsilon(r)|)|X^\varepsilon(r) - x| \\
&\quad + C_L|D_{xx}^2\varphi(X^\varepsilon(r), y) - D_{xx}^2\varphi(x,y)|(1 + |x| + |v^\varepsilon(r)|)^2
\end{aligned}$$

give an estimation for the rest

$$
\begin{aligned}
\theta(\varepsilon) = & \int_0^\varepsilon E\left[\int_0^{\tau_s^\varepsilon} < D_x\varphi(x,y), b(x, v^\varepsilon(r)) > dr\right] ds \\
& - \int_0^\varepsilon E\left[\int_0^{\tau_s^\varepsilon} < D_x\varphi(X^\varepsilon(r),y), b(X^\varepsilon(r), v^\varepsilon(r)) > dr\right] ds \\
& + \int_0^\varepsilon E\left[\int_0^{\tau_s^\varepsilon} \frac{1}{2} Tr\left[D_{xx}^2\varphi(x,y)\sigma\sigma^*(x, v^\varepsilon(r))\right] dr\right] ds \\
& - \int_0^\varepsilon E\left[\int_0^{\tau_s^\varepsilon} \frac{1}{2} Tr\left[D_{xx}^2\varphi(X^\varepsilon(r),y)\sigma\sigma^*(X^\varepsilon(r), v^\varepsilon(r))\right] dr\right] ds.
\end{aligned} \tag{27}
$$

Indeed, in order to evaluate $\theta(\varepsilon)$, we first apply Lemma 9 with $q = 2$ to obtain, for any $\varepsilon > 0$ small enough, that

$$
\int_0^\varepsilon \int_0^s E|X^\varepsilon(r) - x| drds \le C \int_0^\varepsilon \int_0^s r^{\frac{p-2}{2p}} dr = C'\varepsilon^{\frac{5p-2}{2p}}. \tag{28}
$$

Here C and C' are constants independent of ε.

For estimating $\int_0^\varepsilon \int_0^s E[|X^\varepsilon(r) - x||v^\varepsilon(r)|]drds$, we use Hölder's inequality and write

$$
\begin{aligned}
& \int_0^\varepsilon \int_0^s E[|X^\varepsilon(r) - x||v^\varepsilon(r)|]drds \\
\le & \left(\int_0^\varepsilon \int_0^s E[|X^\varepsilon(r) - x|^a]drds\right)^{\frac{1}{a}} \left(\int_0^\varepsilon \int_0^s E[|v^\varepsilon(r)|^{\frac{a}{a-1}}]drds\right)^{\frac{a-1}{a}}.
\end{aligned} \tag{29}
$$

Consequently, Lemma 9 gives $(\int_0^\varepsilon \int_0^s E[|X^\varepsilon(r) - x|^a]drds)^{\frac{1}{a}} \le C'_a\varepsilon^{\frac{3p-a}{ap}}$. On the other hand, from (13) we have

$$
\left(\int_0^\varepsilon \int_0^s E[|v^\varepsilon(r)|^{\frac{a}{a-1}}]drds\right)^{\frac{a-1}{a}} \le C'_a\varepsilon^{\frac{2ap-2p-a}{ap}}. \tag{30}
$$

allowing us to obtain an estimation for the first two terms in (27).

Applying an analogous method to estimate the difference of the terms involving $D_{xx}^2\varphi$ in (27) leads us to the conclusion that, for some $\nu > 0$,

$$
\theta(\varepsilon) \le C_a\varepsilon^{2+\nu}, \tag{31}
$$

for all $\varepsilon > 0$ small enough. We now substitute (24) and (31) in (19). This yields

$$
\begin{aligned}
&C_x e^C \frac{1}{\sqrt{C}}\varepsilon^{\frac{5}{2}} + C_x \varepsilon^{\frac{(6+a)p-a-4}{4p}} + (d_K^2(x)\wedge 1)\varepsilon\left(e^{(L^2+2L)\varepsilon}-1\right)\\
&\geq \int_0^\varepsilon E[\int_0^{\tau_s^\varepsilon} \mathcal{L}_{(x,v^\varepsilon(\cdot))}\varphi_y(x)dr]ds - C_a\varepsilon^{2+\nu} \qquad (32)\\
&\geq \int_0^\varepsilon E[\tau_s^\varepsilon]ds \times \inf_{v\in\mathbb{R}^k} \mathcal{L}_{(x,v)}\varphi_y(x) - C_a\varepsilon^{2+\nu}.
\end{aligned}
$$

Moreover, it can be easily seen that

$$
s \geq E[\tau_s^\varepsilon] \geq s - s\left(P\{\sup_{t\leq s}|X^\varepsilon(t)-x|>1\} + P\{\sup_{t\leq s}|\widetilde{X}^\varepsilon(t)-\widetilde{x}|>1\}\right).
$$

An argument similar to the one exploited to estimate $P(\tau^\varepsilon < \varepsilon)$ allows to prove that $s \geq E[\tau_s^\varepsilon] \geq s - C_a s^{1+\frac{ap-2a}{2p}}$, so that we may deduce

$$
\begin{aligned}
&C_x e^C \frac{1}{\sqrt{C}}\varepsilon^{\frac{5}{2}} + C_x \varepsilon^{\frac{(6+a)p-a-4}{4p}} + C_a\varepsilon^{2+\nu} + (d_K^2(x)\wedge 1)\varepsilon\left(e^{(L^2+2L)\varepsilon}-1\right)\\
&\geq (\frac{\varepsilon^2}{2} - C'_a\varepsilon^{2+\frac{ap-2a}{2p}}) \inf_{v\in\mathbb{R}^k} \mathcal{L}_{(x,v)}\varphi_y(x).
\end{aligned}
$$

We recall that the constants C_x, C_a, C'_a are independent of ε. Thus, we may divide by $\frac{\varepsilon^2}{2}$ and let $\varepsilon \to 0$. This yields

$$
\begin{aligned}
\inf_{v\in\mathbb{R}^k} \mathcal{L}_{(x,v)}\varphi_y(x) &\leq (L^2+2L)(d_K^2(x)\wedge 1)\\
&\leq CF(x,y) - d_K^2(x)\wedge 1 - d_{B_l}(y). \qquad (33)
\end{aligned}
$$

It now suffices to use the fact that $D_y\varphi(x,y) = 0$ and $F(x,y) = \varphi(x,y)$ to conclude

$$
\inf_{v\in\mathbb{R}^k}\left\{\mathcal{L}_{(x,v)}\varphi_y(x) + D_y\varphi(x,y)f(x,y,v)\right\} - CF(x,y) + d_K^2(x)\wedge 1 + d_{B_l}(y) \leq 0.
$$

Thus, the proof of **(b)** is achieved.

c) Let us now suppose $D_y\varphi(x,y) > 0$. Then, in particular, we can find an $\eta > 0$ such that, for any $(x',y') \in A_\eta = \{(x',y') \in \mathbb{R}^n \times \mathbb{R} : |x'-x| + |y'-y| \leq 2\eta\}$ the inequality $D_y\varphi(x',y') > \delta$ holds true for $\delta = \frac{1}{2}D_y\varphi(x,y) > 0$.

We now fix an arbitrarily small $\varepsilon > 0$. Then there exists an admissible control $v^\varepsilon(\cdot)$ satisfying

$$
J(\widetilde{x}, l; v^\varepsilon(\cdot)) \leq \varepsilon^5,
$$

where $\widetilde{x} = \pi_K(x)$.

As before we use the notations

$$
\begin{aligned}
&X^{\varepsilon}(t) = X^{x,v^{\varepsilon}(\cdot)}(t),\ \widetilde{X}^{\varepsilon}(t) = X^{\widetilde{x},v^{\varepsilon}(\cdot)}(t),\\
&Y^{\varepsilon}(t) = Y^{\widetilde{x},y,v^{\varepsilon}(\cdot)}(t),\ \widetilde{Y}^{\varepsilon}(t) = Y^{\widetilde{x},l,v^{\varepsilon}(\cdot)}(t),\ \text{for } t \geq 0.
\end{aligned}
$$

Now, observing that $D_y\varphi(x,y) > 0$ implies that $y \geq l$, we deduce that

$$
E\int_0^{\infty} e^{-Cs}\int_0^s f(\widetilde{X}^{\varepsilon},\widetilde{Y}^{\varepsilon}(r),v^{\varepsilon}(r))drds \leq \varepsilon^5,
$$

and Fubini's theorem yields

$$
E\int_0^{\infty} e^{-Cr} f(\widetilde{X}^{\varepsilon},\widetilde{Y}^{\varepsilon}(r),v^{\varepsilon}(r))dr \leq C\varepsilon^5 \tag{34}
$$

Let

$$
\begin{gathered}
\tau^{\varepsilon} = \inf\{t \geq 0 : |X^{\varepsilon}(t) - x| \vee |\widetilde{X}^{\varepsilon}(t) - \widetilde{x}| \vee |\widetilde{Y}^{\varepsilon}(t) - l| > \eta/2\},\\
\tau_s^{\varepsilon} = s \wedge \tau^{\varepsilon},\ 0 \leq s \leq 1.
\end{gathered}
$$

Then, from our assumption (17) that $F - \varphi$ achieves a global minimum at (x,y), we have:

$$
\int_0^{\varepsilon} E\left[F(X^{\varepsilon}(\tau_s^{\varepsilon}),Y^{\varepsilon}(\tau_s^{\varepsilon})) - F(x,y)\right]ds \geq \int_0^{\varepsilon} E\left[\varphi(X^{\varepsilon}(\tau_s^{\varepsilon}),Y^{\varepsilon}(\tau_s^{\varepsilon})) - \varphi(x,y)\right]ds.
$$

In order to estimate the left side, we remark that, due to the Lipschitz property of f and the nonnegativity of $f(x',\cdot,v')$ on $[l,+\infty)$, for all $x' \in \mathbb{R}^n$, $v' \in U$,

$$
\begin{aligned}
&E\left[d_{B_l}(Y^{\varepsilon}(\tau_s^{\varepsilon})) - d_{B_l}(y)\right] \leq E|Y^{\varepsilon}(\tau_s^{\varepsilon}) - y|\\
&\leq E\left[\int_0^{\tau_s^{\varepsilon}} f(\widetilde{X}^{\varepsilon}(r),Y^{\varepsilon}(r),v^{\varepsilon}(r))dr\right]\\
&\leq E\left[\int_0^{\tau_s^{\varepsilon}} f(\widetilde{X}^{\varepsilon}(r),\widetilde{Y}^{\varepsilon}(r),v^{\varepsilon}(r))dr\right] + LE\left[\int_0^{\tau_s^{\varepsilon}} |Y^{\varepsilon}(r) - \widetilde{Y}^{\varepsilon}(r)|]dr\right]\\
&\leq Ed_{B_l}(\widetilde{Y}^{\varepsilon}(\tau_s^{\varepsilon})) + LE\left[\int_0^{\tau_s^{\varepsilon}} |Y^{\varepsilon}(r) - \widetilde{Y}^{\varepsilon}(r)|dr\right],
\end{aligned}
$$

and

$$
E\left[|Y^{\varepsilon}(r) - \widetilde{Y}^{\varepsilon}(r)|\right] \leq d_{B_l}(y) + L\int_0^r E\left[|Y^{\varepsilon}(u) - \widetilde{Y}^{\varepsilon}(u)|\right]du,
$$

for all $s,r \geq 0$. Then, taking into account that $d_{B_l}(\widetilde{Y}^{\varepsilon}(\tau_s^{\varepsilon})) \leq 1$ and $\widetilde{Y}^{\varepsilon}(\tau_s^{\varepsilon}) \leq \widetilde{Y}^{\varepsilon}(s)$, $s \geq 0$ (recall that $f(x',y',v') \geq 0$ for $y' \geq l$), we get

$$
E[d_{B_l}(Y^{\varepsilon}(\tau_s^{\varepsilon})) - d_{B_l}(y)] \leq E\sqrt{d_{B_l}(\widetilde{Y}^{\varepsilon}(s))} + 2Lsd_{B_l}(y),\ s \geq 0.
$$

Consequently,

$$\begin{aligned}
&\int_0^\varepsilon E\left[F(X^\varepsilon(\tau_s^\varepsilon),Y^\varepsilon(\tau_s^\varepsilon))-F(x,y)\right]ds\\
&\le C_x\int_0^\varepsilon E[d_K(\widetilde{X}^\varepsilon(s))\wedge 1+\sqrt{d_{B_l}(\widetilde{Y}^\varepsilon(s)}]ds\\
&+C_x\int_0^\varepsilon E[|\widetilde{X}^\varepsilon(s)-\widetilde{X}^\varepsilon(\tau_s^\varepsilon)|1_{\{\tau^\varepsilon<\varepsilon\}}]ds+\int_0^\varepsilon E\left[|X^\varepsilon(\tau_s^\varepsilon)-\widetilde{X}^\varepsilon(\tau_s^\varepsilon)|^2\wedge 1\right]ds\\
&-(d_K^2(x)\wedge 1)\varepsilon+L\varepsilon^2 d_{B_l}(y). \qquad (35)
\end{aligned}$$

We observe that

$$\begin{aligned}
P\{\sup_{t\le\varepsilon}|\widetilde{Y}^\varepsilon(t)-l|\ge\eta\}&\le\frac{1}{\eta}E[\sup_{t\le\varepsilon}|\widetilde{Y}^\varepsilon(t)-l|]\\
&\le\frac{1}{\eta}E\int_0^\varepsilon f(\widetilde{X}^\varepsilon(r),\widetilde{Y}^\varepsilon(r),v^\varepsilon(r))dr\le D\varepsilon^5,
\end{aligned}$$

where D is a constant independent of ε (see (32)). This allows to obtain, as in Lemma 8,

$$P(\tau^\varepsilon<\varepsilon)\le A\varepsilon^{\frac{ap-a}{2p}}.$$

Therefore, as for the case $D_y\varphi(x,y)=0$,

$$\begin{aligned}
\int_0^\varepsilon E\left[F(X^\varepsilon(\tau_s^\varepsilon),Y^\varepsilon(\tau_s^\varepsilon))-F(x,y)\right]ds&\le(d_K^2(x)\wedge 1)\varepsilon(e^{(L^2+2L)\varepsilon}-1)\\
&+D\varepsilon^{2+\nu}+L\varepsilon^2 d_{B_l}(y), \qquad (36)
\end{aligned}$$

where $\nu>0$ is some constant independent of ε.

To end the proof of **(c)** we still have to provide a lower bound of the expression $\int_0^\varepsilon E[\varphi(X^\varepsilon(\tau_s^\varepsilon),Y^\varepsilon(\tau_s^\varepsilon)-\varphi(x,y)]ds$. With the notation

$$\begin{aligned}
\mathcal{L}_{(x',x'',y',v)}u(x',y')&=<D_xu(x',y'),b(x',v)>+D_yu(x',y')f(x'',y',v)\\
&+\frac{1}{2}Tr[D^2_{xx}u(x',y')\sigma\sigma^*(x',v)],
\end{aligned}$$

for $(x',y')\in\mathbb{R}^n\times\mathbb{R}$, $v\in\mathbb{R}^k$ and $u\in C^2$. We have from Itô's formula that

$$\begin{aligned}
&\int_0^\varepsilon E\left[\varphi(X^\varepsilon(\tau_s^\varepsilon),Y^\varepsilon(\tau_s^\varepsilon))-\varphi(x,y)\right]ds \qquad (37)\\
&=\int_0^\varepsilon E\left[\int_0^{\tau_s^\varepsilon}\mathcal{L}_{(X^\varepsilon(r),\widetilde{X}^\varepsilon(r)Y^\varepsilon(r),v^\varepsilon(r))}\varphi(X^\varepsilon(r),\widetilde{X}^\varepsilon(r),Y^\varepsilon(r))dr\right]ds.
\end{aligned}$$

We remark that since $D_y\varphi(x,y) \geq \delta > 0$, we have

$$c \overset{not}{=} \inf_{v\in\mathbb{R}^k} \mathcal{L}_{(x,\widetilde{x},y,v)}\varphi(x,y) > -\infty.$$

Moreover, recalling that $A_\eta = \{(x',y') \in \mathbb{R}^n \times \mathbb{R} : |x'-x| + |y'-y| \leq 2\eta\}$ and $B_\eta = \{|x''-\widetilde{x}| \leq \eta\}$ are compact sets, we can find a constant $D > 0$ (independent of ε) such that $|D_x\varphi(x',y')| \leq D$ and $|D^2_{xx}\varphi(x',y')| \leq D$ on A_η. Obviously,

$$\lim_{|v|\to\infty} \left\{ -DL(1+\eta+|v|) + \delta|v|^p - \delta \sup_{x''\in B_\eta} \beta(x'') - DL^2(1+\eta+|v|)^2 \right\} = \infty.$$

Consequently, we can find a constant $\widetilde{D}$ such that, for any $(x',y') \in A_\eta$, $x'' \in B_\eta$ and any $|v| > \widetilde{D}$

$$\mathcal{L}_{(x',x'',y',v)}u(x',y') > c.$$

We return to (37) and, with the help of the above estimate, we can write

$$\begin{aligned} &\int_0^\varepsilon E\left[\varphi(X^\varepsilon(\tau_s^\varepsilon), Y^\varepsilon(\tau_s^\varepsilon)) - \varphi(x,y)\right] ds \geq E\left[\int_0^\varepsilon \int_0^{\tau_s^\varepsilon} c 1_{|v^\varepsilon(r)|>\widetilde{D}} dr ds\right] \\ &+E\left[\int_0^\varepsilon \int_0^{\tau_s^\varepsilon} \mathcal{L}_{(X^\varepsilon(r),\widetilde{X}^\varepsilon(r),Y^\varepsilon(r),v^\varepsilon(r))}(X^\varepsilon(r),Y^\varepsilon(r)) 1_{|v^\varepsilon(r)|\leq\widetilde{D}} dr ds\right]. \end{aligned}$$

For the last term we have

$$\begin{aligned} &E\left[\int_0^\varepsilon \int_0^{\tau_s^\varepsilon} \mathcal{L}_{(X^\varepsilon(r),\widetilde{X}^\varepsilon(r),Y^\varepsilon(r),v^\varepsilon(r))}(X^\varepsilon(r),Y^\varepsilon(r)) 1_{|v^\varepsilon(r)|\leq\widetilde{D}} dr ds\right] \\ &= E\left[\int_0^\varepsilon \int_0^{\tau_s^\varepsilon} \mathcal{L}_{(x,\widetilde{x},y,v^\varepsilon(r))}(x,y) 1_{|v^\varepsilon(r)|\leq\widetilde{D}} dr ds\right] - \theta(\varepsilon) \\ &\geq E\left[\int_0^\varepsilon \int_0^{\tau_s^\varepsilon} c 1_{|v^\varepsilon(r)|\leq\widetilde{D}} dr ds\right] - \theta(\varepsilon), \end{aligned}$$

where $\theta(\varepsilon)$ is defined by the above equality. Therefore, from the definition of c,

$$\begin{aligned} &\int_0^\varepsilon E\left[\varphi(X^\varepsilon(\tau_s^\varepsilon), Y^\varepsilon(\tau_s^\varepsilon)) - \varphi(x,y)\right] ds \\ &\geq \int_0^\varepsilon E[\tau_s^\varepsilon] ds \times \inf_{v\in\mathbb{R}^k} \mathcal{L}_{(x,\widetilde{x},y,v)}\varphi(x,y) - \theta(\varepsilon). \end{aligned} \tag{38}$$

When evaluating $\theta(\varepsilon)$ we use (28) and (29) which reduces the problem to estimating terms of the form

$$E\int_0^\varepsilon \int_0^s |Y^\varepsilon(r)-y||v^\varepsilon(r)|^q 1_{|v^\varepsilon(r)|\le \widetilde{D}} drds$$

which can obviously be estimated from above by

$$\begin{aligned} E\int_0^\varepsilon \int_0^s |Y^\varepsilon(r)-y||v^\varepsilon(r)|^q 1_{|v^\varepsilon(r)|\le \widetilde{D}} drds &\le \widetilde{D}^q E\int_0^\varepsilon \int_0^s |Y^\varepsilon(r)-y| drds \\ &\le D\varepsilon^7. \end{aligned}$$

First, combining (36) and (38), then dividing by $\frac{\varepsilon^2}{2}$ and, finally, taking the limit as $\varepsilon \to 0$ we obtain

$$\begin{aligned} 0 \le & \sup_{v\in\mathcal{A}}\{< D_x\varphi(x,y), c(x,v) > + D_y\varphi(x,y) f(\widetilde{x},y,v) \\ & + \frac{1}{2} Tr[D^2_{xx}\varphi(x,y)\sigma\sigma^*(x,v)]\} + C\varphi(x,y) - d_K^2(x)\wedge 1 - d_{B_l}(y). \end{aligned}$$

Finally, it suffices to recall the assumption $f(\widetilde{x},y,v) = f(x,y,v)$ and the conclusion follows.

The proof that (i) implies (ii) is now complete.

For the converse, suppose that F is a viscosity supersolution for (12). Then, using the fact that V is a viscosity subsolution of (12) and that F and V enjoy the Lipschitz property, we get $V(x) \le F(x)$ for all $x \in \mathbb{R}^n$. In particular, it is obvious that $V(x) = 0$ for $x \in K$.□

We now turn our attention to the proof of Lemmata 8 and 9.

PROOF. (of the Lemma 8). By the definition of the hitting time τ^ε, it is straight-forward that

$$\begin{aligned} P(\tau^\varepsilon < \varepsilon) &\le P(\sup_{t\le\varepsilon}|X^\varepsilon(t)-x|>1) + P(\sup_{t\le\varepsilon}|\widetilde{X}^\varepsilon(t)-\widetilde{x}|>1) \\ &\le E[\sup_{t\le\varepsilon}|X^\varepsilon(t)-x|^a] + E[\sup_{t\le\varepsilon}|\widetilde{X}^\varepsilon(t)-\widetilde{x}|^a]. \end{aligned} \tag{39}$$

We also notice that the following inequality holds true:

$$
\begin{aligned}
|X^\varepsilon(t)-x| \leq & \int_0^t (L+L|x|+L|X^\varepsilon(r)-x|+L|v^\varepsilon(r)|)dr \\
& + \left| \int_0^t \sigma(X^\varepsilon(r), v^\varepsilon(r))dW(r) \right|.
\end{aligned}
$$

Moreover, there exists a constant C_a (depending on a, x and L but not on ε) such that

$$
\begin{aligned}
|X^\varepsilon(t)-x|^a \leq & C_a t^a + C_a L^a \left(\int_0^t |X^\varepsilon(r)-x|dr \right)^a + C_a L^a \left(\int_0^t |v^\varepsilon(r)|dr \right)^a \\
& + C_a \left| \int_0^t \sigma(X^\varepsilon(r), v^\varepsilon(r))dW(r) \right|^a.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
E\left[\sup_{t\leq\varepsilon} |X^\varepsilon(t)-x|^a\right] \leq & C_a\varepsilon^a + C_a\varepsilon^a E\left[\sup_{t\leq\varepsilon} |X^\varepsilon(t)-x|^a\right] \\
& + C_a E\left[\left(\int_0^\varepsilon |v^\varepsilon(r)|dr \right)^a\right] \\
& + C_a E\left[\sup_{t\leq\varepsilon} \left| \int_0^t \sigma(X^\varepsilon(r), v^\varepsilon(r))dW(r) \right|^a\right].
\end{aligned} \tag{40}
$$

Further, applying Hölder's inequality we find that

$$
\begin{aligned}
E\left[\left(\int_0^\varepsilon |v^\varepsilon(r)|dr \right)^a\right] \leq & \varepsilon^{a-1} E\left[\int_0^\varepsilon |v^\varepsilon(r)|^a dr\right] \\
\leq & \varepsilon^{a-1}\varepsilon^{\frac{p-a}{p}} \left(E\left[\int_0^\varepsilon |v^\varepsilon(r)|^p dr\right]\right)^{\frac{a}{p}}.
\end{aligned}
$$

The choice of v^ε and the Remark 7 on the $1-$optimal controls guarantee that $v^\varepsilon(\cdot) \in \mathcal{A}^1(\widetilde{x}, \widetilde{y})$. Consequently, for some constant C'_a only depending on a, $\widetilde{x}$ and $\widetilde{y}$,

$$
E\left[\left(\int_0^\varepsilon |v^\varepsilon(r)|dr \right)^a\right] \leq C'_a \varepsilon^{\frac{ap-a}{p}}.
$$

We return to the inequality (40) that can now be written in the following form

$$
\begin{aligned}
E\left[\sup_{t\leq\varepsilon} |X^\varepsilon(t)-x|^a\right] \leq & C_a\varepsilon^a + C_a\varepsilon^a E\left[\sup_{t\leq\varepsilon} |X^\varepsilon(t)-x|^a\right] \\
& + C_a C'_a \varepsilon^{\frac{ap-a}{p}} \\
& + C_a E\left[\sup_{t\leq\varepsilon} \left| \int_0^t \sigma(X^\varepsilon(r), v^\varepsilon(r))dW(r) \right|^a\right].
\end{aligned}
$$

Finally, we apply the Burkholder-Davis-Gundy inequality to the last term on the right side to obtain

$$E\left[\sup_{t\le\varepsilon}|X^\varepsilon(t)-x|^a\right]\le C_a(\varepsilon^a+\varepsilon^{\frac{a}{2}})+C_a(\varepsilon^a+\varepsilon^{\frac{a}{2}})E\left[\sup_{t\le\varepsilon}|X^\varepsilon(t)-x|^a\right]$$
$$+C_a\varepsilon^{\frac{ap-a}{p}}+C_a\varepsilon^{\frac{ap-2a}{2p}},$$

where C_a is again a constant independent of ε. The conclusion of our Lemma now follows easily.□

PROOF. (of the Lemma 9). Applying Itô's formula to $|X^\varepsilon(s)-x|^q$ we obtain

$$E\left[|X^\varepsilon(s)-x|^q\right]\le qLE\left[\int_0^s|X^\varepsilon(r)-x|^{q-1}(1+|x|+|X^\varepsilon(r)-x|+|v^\varepsilon(r)|)dr\right]$$
$$+\frac{q(q-1)}{2}L^2E\left[\int_0^s|X^\varepsilon(r)-x|^{q-2}(1+|x|+|X^\varepsilon(r)-x|+|v^\varepsilon(r)|)^2dr\right].$$

We recall that, for any $b,c>0$, we have

$$b^{q-1}c\le\frac{q-1}{q}b^q+\frac{1}{q}c^q \text{ and } b^{q-2}c^2\le\frac{q-2}{q}b^q+\frac{2}{q}c^q.$$

With these inequalities and Gronwall's Lemma we obtain, for $s\le1$,

$$E\left[|X^\varepsilon(s)-x|^q\right]\le C_q(1+|x|)^qs+C_qE\left[\int_0^s|v^\varepsilon(r)|^qdr\right].$$

Finally,

$$E\left[\int_0^s|v^\varepsilon(r)|^qdr\right]\le E\left[\int_0^s|v^\varepsilon(r)|^pdr\right]^{\frac{q}{p}}\times s^{\frac{p-q}{p}}.$$

The conclusion of our Lemma follows easily.□

4 An illustrating example

We now concentrate our attention on a simple but illustrating example. We consider the case of an unbounded cylinder in $\mathbb{R}^n$ and calculate explicitly the necessary and sufficient condition for ε-viability given by Theorem 6. We exhibit, for a particular choice of the coefficient functions, a necessarily unbounded feedback control process which cannot be managed by the compact control state space framework.

Example 10 *Let $\widehat{\pi}:\mathbb{R}^n\longrightarrow\mathbb{R}^n$ be the linear projection on the $n-1$ first coordinates*

$$\widehat{\pi}(x)=(x_1,x_2,\cdots,x_{n-1},0),\ \textit{for } x=(x_1,x_2,\cdots,x_n)\in\mathbb{R}^n$$

and $\pi_n:\mathbb{R}^n\longrightarrow\mathbb{R}$ the projection on the last component, $\pi_n((x_1,x_2,\cdots,x_n))=x_n$.

As set K we consider the cylinder $K = \{x \in \mathbb{R}^n : |\widehat{\pi}(x)| \leq R\}$.

Then,

$$d_K(x) = (|\widehat{\pi}(x)| - R)^+,$$

$$D_x d_K^2(x) = \begin{cases} 2\frac{|\widehat{\pi}(x)|-R}{|\widehat{\pi}(x)|}\widehat{\pi}(x), & \text{if } |\widehat{\pi}(x)| > R, \\ 0, & \text{if } |\widehat{\pi}(x)| \leq R, \end{cases}$$

and

$$D_{xx}^2 d_K^2(x) = \begin{cases} 2\frac{|\widehat{\pi}(x)|-R}{|\widehat{\pi}(x)|}\widehat{I} + 2\frac{R}{|\widehat{\pi}(x)|^3}\widehat{\pi}(x) \otimes \widehat{\pi}(x) & , \text{ if } |\widehat{\pi}(x)| > R, \\ 0, & \text{if } |\widehat{\pi}(x)| < R, \end{cases}$$

where $\widehat{I} = \begin{pmatrix} I_{n-1} & 0 \\ 0 & 0 \end{pmatrix}$, and I_{n-1} denotes the unit matrix of type $(n-1)\times(n-1)$.

Proposition 11 *Suppose that $|\sigma(\cdot, v)\widehat{I}|^2$ is Lipschitz uniformly with respect to $v \in \mathbb{R}^k$. Then K enjoys the ε-viability property for the equation (9) if and only if the following conditions hold true simultaneously:*

(a) For all $x \in \mathbb{R}^n$ with $|\widehat{\pi}(x)| = R$, there exists $v \in \mathbb{R}^k$ satisfying

$$\begin{cases} (1)\ \sigma^*(x, v)\widehat{\pi}(x) = 0; \\ (2)\ 2 < \widehat{\pi}(x), b(x, v) > +|\sigma^*(x, v)\widehat{I}|^2 \leq 0; \\ (3)\ f(x, l, v) = 0. \end{cases}$$

(b) For all $x \in \mathbb{R}^n$ such that $|\widehat{\pi}(x)| < R$, there exists $v \in \mathbb{R}^k$ satisfying

$$f(x, l, v) = 0.$$

Remark 12 *For the particular case $f(x, y, v) = ||v|^p - |\pi_n(x)||$, we notice that every feedback control $v = v(x)$ that satisfies (a) must also satisfy $|v(x)| = |\pi_n(x)|^{\frac{1}{p}}$.*

Let us emphasize that the feedback control process is necessarily unbounded in $x \in \mathbb{R}^n$. This shows that the above example cannot be covered by the existing results in the literature on viability of controlled stochastic systems with bounded control state space.

PROOF. (of Proposition 11). Let us first suppose that F is a viscosity supersolution of (12).

For any $|x| > R$ and any $y > l$, the fact that F is a viscosity supersolution of (12) implies:

$$\inf_{v\in\mathbb{R}^k}\left\{2\frac{|\widehat{\pi}(x)|-R}{|\widehat{\pi}(x)|}\langle\widehat{\pi}(x),b(x,v)\rangle+\frac{|\widehat{\pi}(x)|-R}{|\widehat{\pi}(x)|}|\sigma^*(x,v)\widehat{I}|^2\right.$$
$$\left.+\frac{R}{|\widehat{\pi}(x)|^3}|\sigma^*(x,v)\widehat{\pi}(x)|^2+f(x,y,v)\right\}$$
$$\leq(C-1)[y-l+(|\widehat{\pi}(x)|-R)^2\wedge 1].$$

Let us fix an arbitrary $x_0 \in \mathbb{R}^n$ such that $|\widehat{\pi}(x_0)| = R$. As $f(x, y, v) \geq |v|^p - \beta(x)$, we deduce that, for any $x \in \mathbb{R}^n$ with $|x| > R$, $y > l$, and any $\varepsilon > 0$ there is some $v^{\varepsilon,x,y} \in \mathbb{R}^k$ such that

$$2\frac{|\widehat{\pi}(x)|-R}{|\widehat{\pi}(x)|}<\widehat{\pi}(x),b(x,v^{\varepsilon,x,y})>+\frac{|\widehat{\pi}(x)|-R}{|\widehat{\pi}(x)|}|\sigma^*(x,v^{\varepsilon,x,y})\widehat{I}|^2$$
$$+\frac{R}{|\widehat{\pi}(x)|^3}|\sigma^*(x,v^{\varepsilon,x,y})\widehat{\pi}(x)|^2+f(x,y,v^{\varepsilon,x,y})$$
$$\leq(C-1)[y-l+(|\widehat{\pi}(x)|-R)^2\wedge 1]+\varepsilon,$$

and the family $\{|v^{\varepsilon,x,y}|,\ (\varepsilon, x, y) \in [0, 1] \times B_1(x_0) \times [l, l + 1]\}$ is uniformly bounded.

Thus, by choosing sequences $(x_m)_{m\geq 1} \subset B_1(x_0)$ with $x_m \to x_0$ and $|\widehat{\pi}(x_m)| > R$, $y_m = l + (|\widehat{\pi}(x_m)| - R)^2 \to l$, and $1 \geq \varepsilon_m = (|\widehat{\pi}(x_m)| - R)^2 \searrow 0$, and we obtain the existence of a bounded sequence $v^m = v^{\varepsilon_m,x_m,y_m}$ such that

$$\frac{1}{|\widehat{\pi}(x_m)|}\left(2<\widehat{\pi}(x_m),b(x_m,v^m)>+|\sigma^*(x_m,v^m)\widehat{I}|^2\right)$$
$$+\frac{1}{|\widehat{\pi}(x_m)|-R}\left(\frac{R}{|\widehat{\pi}(x_m)|^3}|\sigma^*(x_m,v^m)\widehat{\pi}(x_m)|^2+f(x_m,y_m,v^m)\right)$$
$$\leq(2C-1)(|\widehat{\pi}(x_m)|-R).$$

Moreover, there exists a subsequence (still denoted v^m) and some $v \in \mathbb{R}^k$ such that $\lim_m v^m = v$. Consequently, letting $m \to \infty$ in the above inequality we obtain

$$\frac{1}{R^2}|\sigma^*(x_0,v)\widehat{\pi}(x_0)|^2+f(x_0,l,v)\leq 0$$

and

$$2<\widehat{\pi}(x_0),b(x_0,v)>+|\sigma^*(x_0,v)\widehat{I}|^2\leq 0.$$

Finally, since $f(x_0, l, v) \geq 0$ we get condition *(a)* of the statement.

To prove the second condition in the statement, we remark that for any $|\widehat{\pi}(x)| < R$ and any $y > l$, the fact that F is a viscosity supersolution for

(12) implies:

$$\inf_{v\in\mathbb{R}^k} f(x,y,v) \le (C-1)(y-l).$$

We fix $x \in \mathbb{R}^n$ such that $|\widehat{\pi}(x)| < R$. As $f(x,y,v) \ge |v|^p - \beta(x)$, we deduce that, for any $y > l$, and any $\varepsilon > 0$ there exists $|v^{\varepsilon,y}|$ such that

$$f(x,y,v^{\varepsilon,y}) \le (C-1)(y-l)+\varepsilon$$

and $|v^{\varepsilon,y}|$ is uniformly bounded on $(\varepsilon,y)\in[0,1]\times[l,l+1]$.

We choose $y_m \searrow l$ and $\varepsilon_m \to 0$, and letting $m\to\infty$ yields the second assertion in the statement.

We now prove the sufficiency of the conditions *(a)* and *(b)*.

For any $x \in K$ and any $y > l$, the Lipschitz property of f yields

$$f(x,y,v) \le f(x,l,v) + L(y-l),$$

and we may choose $v \in \mathbb{R}^k$ satisfying $f(x,l,v)=0$ to get

$$f(x,y,v) \le (C-1)(y-l).$$

Let us now consider $x \in \mathbb{R}^n$ such that $|\widehat{\pi}(x)| > R$. Then, obviously,

$$\pi_K(x_1,x_2,\cdots,x_n) = \left(\frac{R}{|\widehat{\pi}(x)|}x_1, \frac{R}{|\widehat{\pi}(x)|}x_2, \cdots, \frac{R}{|\widehat{\pi}(x)|}x_{n-1}, x_n\right).$$

Using this remark and the Lipschitz property of the coefficient functions b and σ, we obtain that the following inequalities hold true for all $x\in\mathbb{R}^n$ such that $R < |x| \le R+1$, $y > l$ and all $v \in \mathbb{R}^k$:

$$\frac{1}{|\widehat{\pi}(x)|}\left\langle \widehat{\pi}(x), b(x,v)\right\rangle - \frac{1}{R}\left\langle \frac{R}{|\widehat{\pi}(x)|}\widehat{\pi}(x), b(\pi_K(x),v)\right\rangle \le L(|\widehat{\pi}(x)|-R)$$

and

$$\frac{1}{|\widehat{\pi}(x)|}|\sigma^*(x,v)\widehat{I}|^2 - \frac{1}{R}|\sigma^*(\pi_K(x),v)\widehat{I}|^2 \le \frac{1}{|\widehat{\pi}(x)|}\widetilde{L}(|\widehat{\pi}(x)|-R).$$

Moreover, for some $v\in\mathbb{R}^n$ for which *(a)* holds true, we have:

$$\frac{R}{|\widehat{\pi}(x)|^3}|\sigma^*(x,v)\widehat{\pi}(x)|^2 + f(x,y,v) \le L^2(|\widehat{\pi}(x)|-R)^2 + L(y-l).$$

Therefore,

$$
\begin{aligned}
&2\frac{|\widehat{\pi}(x)|-R}{|\widehat{\pi}(x)|}\langle\widehat{\pi}(x),b(x,v)\rangle+\frac{|\widehat{\pi}(x)|-R}{|\widehat{\pi}(x)|}|\sigma^*(x,v)\widehat{I}|^2 \\
&+\frac{R}{|\widehat{\pi}(x)|^3}|\sigma^*(x,v)\widehat{\pi}(x)|^2+f(x,y,v) \\
&\le(|\widehat{\pi}(x)|-R)^2(2L+\widetilde{L})+(C-1)(y-l).
\end{aligned}
\tag{41}
$$

We consider test functions $\varphi\in C^2(\mathbb{R}^n\times\mathbb{R})$ and $(x,y)\in\mathbb{R}^n\times\mathbb{R}$ local minimum for $F-\varphi$ and wish to prove

$$
\begin{aligned}
0\le\sup_{v\in\mathbb{R}^k}&\{-<D_x\varphi(x,y),b(x,v)>-D_y\varphi(x,y)f(x,y,v) \\
&-\frac{1}{2}Tr[D^2_{xx}\varphi(x,y)\sigma\sigma^*(x,v)]\}+CF(x,y)-d^2_K(x)\wedge 1-d_{B_l}(y)\ .
\end{aligned}
\tag{42}
$$

This is obvious enough for $|x|\le R$ or $y<-l$ (see Theorem 6 for the latter). The inequality (41) allows us to deal with points (x,y) such that $R<|x|$ and $l<y$ and condition (b) in the statement gives the result for the remaining case. We obtain that F is a viscosity supersolution for (12). The conclusion follows from Theorem 6.□

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Approximate controllability for linear stochastic differential equations with control acting on the noise

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Abstract

In this paper we study approximate controllability for a linear stochastic differential equation

$$dy(t) = (Ay(t) + Bu(t))\,dt + (Cy(t) + Du(t))\,dW(t),$$

for the case when the control acts also on the noise. This may be considered as a generalization of the work of Buckdahn, Quincampoix and Tessitore where the problem is solved for $D = 0$ and of Peng for D of full rank. We prove, using the dual BSDE and Riccati methods that approximate controllability is equivalent to the local in time viability for a suitable set. Finally, an invariance criterion is given.

Key words: Stochastic Control, Controllability, Controllability under Constraints, Backward Stochastic Differential Equation, Riccati Equation.

Introduction and statement of the main result

Given a complete probability space $(\Omega, \mathcal{F}, P)$ and a standard Brownian motion $(W(t), t \geq 0)$ on this space, we consider the natural complete filtration $(\mathcal{F}_t)_{t\geq 0}$ generated by W. We let $\mathcal{A} = \mathcal{A}(\Omega, \mathcal{F}, P; W)$ be the set of all $(\mathcal{F}_t)$-progressively measurable processes $v(\cdot)$ taking their values in $\mathbb{R}^d$ and such

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that $E\left[\int_0^T |v(s)|^2 ds\right] < \infty$ for all $T > 0$. A process $v(\cdot) \in \mathcal{A}$ will be referred as an admissible control process.

We consider the following linear stochastic differential equation

$$dy(t) = (Ay(t) + Bu(t))\, dt + (Cy(t) + Du(t))\, dW(t),\ 0 \leq t \leq T, \qquad (1)$$

governed by the control process $u(\cdot) \in \mathcal{A}$, where $A, C \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, and $B, D \in \mathcal{L}(\mathbb{R}^d, \mathbb{R}^n)$. The initial condition is

$$y(0) = x \in \mathbb{R}^n. \qquad (2)$$

For all $x \in \mathbb{R}^n$ and all admissible control $u(\cdot)$, the equation (1) admits an unique predictable solution $y(\cdot, x, u)$ with continuous trajectories such that $y(0, x, u) = x$. Furthermore, this solution satisfies, for all $T > 0$,

$$E\left[\sup_{s\in[0,T]} |y(s)|^2\right] < \infty.$$

In [7] and [5], Peng (respectively Liu and Peng) have studied the "exact controllability" and "exact terminal controllability" for (1). They show that exact terminal controllability is equivalent with the condition that D has full rank. Moreover, algebraic conditions of Kalman type give a characterization of exact controllability. The case where the control $u(\cdot)$ does not act on the noise (i.e. $D = 0$) has been studied by Buckdahn, Quincampoix and Tessitore in [4]. Since D is not of full rank, exact terminal controllability must be weakened to "approximate controllability". Algebraic Riccati equation methods are used in [4] in order to obtain a characterization of approximately-controllable stochastic linear equations. This criterion says that, in order to have approximate controllability, the only locally in time viable subset of a certain linear space has to be the trivial set.

The objective of our paper is to extend the results in [4] and [7] to the general case where the control is allowed to act on the noise (i.e. $rank(D) \geq 0$) without necessarily having D of full rank. We give a characterization of approximate controllability using the notion of local in time viability of a linear space $V \subset \mathbb{R}^n$ conditioned to another linear space $U \subset \mathbb{R}^n$(a generalization of the concept of local in time viability (l.i.t.v.)) introduced in [4]). An easily computable criterion is given.

We prove that the study of the linear stochastic differential equation (1) can be reduced to an equation of the form

$$dy(t) = (Ay(t) + B_1u'(t) + B_2u''(t))dt + (Cy(t) + D_1u'(t))dW(t), \qquad (3)$$

where $A, C \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, $B_1, D_1 \in \mathcal{L}(\mathbb{R}^r, \mathbb{R}^n)$, $B_2 \in \mathcal{L}(\mathbb{R}^{d-r}, \mathbb{R}^n)$ and *rank* $D_1 = r$.

The main result of the present paper is

Theorem 1 *We have equivalence between the following assertions:*

1. *The equation (3) is approximately controllable.*

2. *The equation (3) is approximately null controllable.*

3. *The viability kernel of* $Ker\ B_2^*$ *conditioned to* $Ker\ D_1^*$ *is trivial.*

1 Preliminaries

Let us recall the following definition of approximate controllability

Definition 2 *We say that the equation (1) is approximately controllable if, for all* $x \in \mathbb{R}^n$, *all* $T > 0$, *all* $\eta \in L^2(\Omega; \mathcal{F}_T\ ; P; \mathbb{R}^n)$, *and all* $\varepsilon > 0$, *there exists an admissible control* u *such that*

$$E\left[|y(T, x, u) - \eta|^2\right] \leq \varepsilon.$$

Moreover, we say that the equation (1) is approximately null controllable if the above condition holds for the particular case $\eta = 0$.

In order to obtain computable algebraic conditions for approximate controllability in the case where the control is not allowed to act on the noise (i.e. $D = 0$), the authors of [4] use the notion of strict invariance. This concept (slightly modified) had already been used in [9] to obtain, by algebraic Riccati equation methods, a characterization of stochastic linear equations admitting a feedback that stabilizes the system for all noise intensities.

Definition 3 *Given* $m + 1$ *linear operators* $L; M_1; ...; M_m \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, *a linear subspace* $V \subset \mathbb{R}^n$ *is said to be* $(L; M_1; ...; M_m)$*-strictly invariant if* $LV \subset Span\{V; M_1V; ...; M_mV\}$.

We notice that a linear subspace $V \subset \mathbb{R}^n$ is $(L; M_1; ...; M_m)$-strictly invariant if and only if there exist operators $K_1, ..., K_m \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ such that $K_iV \subset V$ and $(L + M_1K_1 + ... + M_mK_m)V \subset V$ or, equivalently, if and only if for all $v \in V$ there exist $w_1, ..., w_m \in V$ such that $Lv + M_1w_1 + ... + M_mw_m \in V$.

Since we allow the control to act on the noise, we have to require a stronger property than strict invariance. Namely, we give the following Definition, which extends the notion of strict invariance used in [4] to establish a characterization of l.i.t.v.

Definition 4 *Given the linear operators $L, M, N \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ and two linear subspaces $V \subset \mathbb{R}^n$ and $U \subset \mathbb{R}^n$, we say that V is $(L; M)$-strictly invariant conditioned to (N, U) if for all $v \in V$ there exists $w \in V$ such that*

$$w - Nv \in U \text{ and } Lv + Mw \in V.$$

Remark 5 *It is easy to see that V is $(L; M)$-strictly invariant conditioned to (N, U) if and only if there exists a linear operator $K \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ such that $KV \subset V$, $(K - N)V \subset U$ and $(L + MK)V \subset V$.*

Remark 6 *For all N it can be easily observed that V is $(L; M)$-strictly invariant conditioned to $(N, \mathbb{R}^n)$ if and only if it is $(L; M)$-strictly invariant (i.e., if $U = \mathbb{R}^n$ is the full space we find the notion of strict invariance).*

Remark 7 *For arbitrary linear subspaces $V, U \subset \mathbb{R}^n$, the largest subspace of V which is $(L; M)$-strictly invariant conditioned to (N, U) can be obtained in at most n iterations by considering the following schema*

$$V_0 = V; \; V_{i+1} = \{v \in V_i : \; M((U + Nv) \cap V_i) \cap (V_i - Lv) \neq \phi\}, \; i \in \mathbb{N}.$$

2 The dual equation

We consider the following backward stochastic differential equation

$$\begin{cases} dp(t) = -\left(A^*p(t) + C^*q(t)\right) dt + q^*(t)dW(t), \; 0 \leq t \leq T, \\ p(T) = \eta \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^n). \end{cases} \tag{4}$$

It has been established in [6] that for all $T > 0$ and all $\eta \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^n)$ (4) admits an unique solution $(p, q) \in M^2(0, T; \mathbb{R}^n) \times M^2(0, T; \mathbb{R}^n)$ (where $M^2(0, T; \mathbb{R}^n)$ stands for the set of $\mathbb{R}^n$-valued processes which are $\mathcal{F}_t$-progressively measurable and square integrable over $\Omega \times [0, T]$ with respect to $P \otimes dt$). Moreover, p has continuous trajectories and $E\left[\sup_{s \in [0,T]} |p(s)|^2\right] < \infty$.

We are able at this point to prove the connection between approximate controllability for (1) and the backward stochastic differential equation (4):

Proposition 8 *The equation (1) is approximately-controllable if and only if, for all $T > 0$, every solution of (4) such that $B^*p(s) + D^*q(s) = 0$, $P - a.s.$,* *for all $s \in [0,T]$ is trivial.*

Moreover, the equation (1) is approximately null controllable if and only if, for all $T > 0$, every solution of (4) such that $B^*p(s) + D^*q(s) = 0$, $P - a.s.$, for all $s \in [0,T]$ satisfies $p(0) = 0$.

PROOF. Let us fix $T \geq 0$. Using Itô's formula for $\langle p(T), y(T,x,u)\rangle$ we have

$$E\left[\langle p(T), y(T,x,u)\rangle\right] - E[\langle p(0), x\rangle] = E\left[\int_0^T \langle B^*p(s) + D^*q(s), u(s)\rangle\, ds\right] \quad (5)$$

We denote by $L^2_{\mathcal{P}}([0,T],\mathbb{R}^d)$ the space of all predictable processes $u : \Omega \times [0,\infty[\longrightarrow \mathbb{R}^d$ satisfying $E\left[\int_0^T |u(s)|^2 ds\right] < \infty$ and consider the linear operator M_T given by

$$M_T : L^2_{\mathcal{P}}([0,T],\mathbb{R}^d) \longrightarrow L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^n), \quad M_T u = y(T,0,u). \qquad (6)$$

It is straightforward that (1) is approximately-controllable if and only if, for all $T > 0$, $M_T\left(L^2_{\mathcal{P}}([0,T],\mathbb{R}^d)\right)$ is dense in $L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^n)$. The equality (5) implies $M_T^*\eta = B^*p + D^*q$. We use the fact that the image of a linear operator is dense if and only if the kernel of its adjoint is trivial and the uniqueness and the continuity of the solution of (4) to get $\eta = 0$ if and only if $p(s) = 0$, $P - a.s.$, for all $s \in [0,T]$ and $q(s) = 0$ $dsdP$-almost everywhere on $[0,T] \times \Omega$.

In order to prove the second assertion in our statement, we introduce the linear operator

$$L_T : \mathbb{R}^n \longrightarrow L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^n), \quad L_T x = y(T,x,0). \qquad (7)$$

It is obvious that approximate null controllability is equivalent to the fact that for all $T > 0$, $L_T[\mathbb{R}^n] \subset \overline{M_T[L^2_{\mathcal{P}}([0,T],\mathbb{R}^d)]}$ (or $Ker\ (M_T^*) \subset Ker\ (L_T^*)$) . To conclude the proof of this second part, we use (5) for $u = 0$ and obtain $L_T^*\eta = p(0)$. The conclusion follows.

In [7], for the case where D is a full rank matrix, the author was able to transform the equation into the equivalent form

$$dy(t) = (Ay(t) + A_1u'(t) + Bu''(t))dt + u'(t)dW(t), \ 0 \leq t \leq T. \qquad (8)$$

Using this idea, for the case where $0 \leq rank\ D = r \leq n$, we get the following linear stochastic differential equation, which is in fact equivalent to (1):

$$dy(t) = (Ay(t) + B_1u'(t) + B_2u''(t))dt + (Cy(t) + D_1u'(t))dW(t),$$

where $A, C \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, $B_1, D_1 \in \mathcal{L}(\mathbb{R}^r, \mathbb{R}^n)$, $B_2 \in \mathcal{L}(\mathbb{R}^{d-r}, \mathbb{R}^n)$ and *rank* $D_1 = r$. Since *rank* $D_1 = r$ we establish the existence of $F \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ solution of $D_1^* F + B_1^* = 0$. In this case, the dual equation becomes

$$\begin{cases} dp(t) = [-(A^* + C^* F)p(t) - C^* q(t)]\, dt + (Fp(t) + q(t))dW(t), \\ p(T) = \eta. \end{cases} \tag{9}$$

From [6], Theorem 4.1, we have that for all $T > 0$ and all $\eta \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^n)$,the backward stochastic differential equation (9) admits an unique solution $(p, q) \in M^2(0, T; \mathbb{R}^n) \times M^2(0, T; \mathbb{R}^n)$ such that p has continuous trajectories and

$$E \left[\sup_{s \in [0,T]} |p(s)|^2 \right] < \infty.$$

As before, using Itô's formula we show that

$$\begin{aligned} E < p(T), y(T) >= E < p(0), y(0) > \\ + E \left[\int_0^T (< B_2^* p(t), u''(t) > + < D_1^* q(t), u'(t) >)\, dt \right], \end{aligned}$$

and, with the same arguments as for the previous Proposition we can prove

Proposition 9 *The equation (3) is approximately-controllable if and only if, for all $T > 0$, every solution of (9) such that $B_2^* p(s) = 0$ and $D_1^* q(s) = 0$, $P - a.s.$, for all $s \in [0, T]$ is trivially reduced to 0.*

Moreover, the equation (3) is approximately null controllable if and only if, for all $T > 0$, every solution of (9) such that $B_2^* p(s) = 0$ and $D_1^* q(s) = 0$, $P - a.s.$, for all $s \in [0, T]$ satisfies $p(0) = 0$.

Approximate controllability property for (3) can be expressed with the help of the following quadratic cost function

$$\begin{aligned} J(\theta, q(\cdot)) = E\{ \int_0^T [< \Pi_{(Ker\ B_2^*)^\perp} p(t, q, \theta), p(t, q, \theta) > \\ + < \Pi_{(Ker\ D_1^*)^\perp} q(t), q(t) >] dt \}. \end{aligned}$$

Indeed, let us consider the Stochastic Linear Quadratic optimal control problem

Problem 10 ***(SLQ)***

For each $\theta \in \mathbb{R}^n$ find an admissible control $\overline{q}(\cdot)$ such that

$$J(\theta, \overline{q}(\cdot)) = \inf J(\theta, q(\cdot)) \overset{not}{=} V(\theta)$$

Definition 11 *The (SLQ) problem is said to be*

(1) solvable at $\theta \in \mathbb{R}^n$ if there exists an admissible control $\overline{q}(\cdot)$ such that $J(\theta, \overline{q}(\cdot)) = V(\theta)$. In this case, $\overline{q}(\cdot)$ is called an optimal control.

(2) pathwise uniquely solvable at $\theta \in \mathbb{R}^n$ if it is solvable at θ and, whenever $q_1(\cdot)$ and $q_2(\cdot)$ are two optimal controls on the same space, it holds

$$P(\{q_1(t) = q_2(t), \text{ for almost every } t \in [0, T]\}) = 1.$$

The following proposition gives the connection between approximate controllability, approximate null controllability and the (SLQ) problem.

Proposition 12 *We have equivalence between the following assertions:*

(i) The equation (3) is approximately-controllable.

(ii) For all $T > 0$, all $\theta \in \mathbb{R}^n$, and all $q \in L^2_{\mathcal{P}}([0, T], Ker\ D_1^*)$ such that $B_2^* p(s, q, \theta) = 0$, $P - a.s..$, for all $s \in [0, T]$ it must hold that $q(s) = 0$ $dsdP$-almost everywhere on $[0, T] \times \Omega$ and $\theta = 0$;

$$\text{(iii)} \begin{cases} \text{a) The (SLQ) problem is pathwise uniquely solvable at } \theta = 0; \\ \text{b) The equation (3) is approximately null controllable.} \end{cases}.$$

PROOF. We only have to prove the equivalence between the last two assertions. Let us suppose that *(ii)* holds true. It is obvious that (SLQ) is solvable at $\theta = 0$ and that $\overline{q}(\cdot) \equiv 0$ is an optimal control process. If we consider $q(\cdot)$ to be another optimal control at $\theta = 0$, we must have

$$J(0, q(\cdot)) = 0.$$

Therefore, we get $q \in L^2_{\mathcal{P}}([0, T], Ker\ D_1^*)$ and $B_2^* p(s, q, \theta) = 0$, $P - a.s.$, for all $s \in [0, T]$. We now use *(ii)* to obtain that $q(s) = 0$ $dsdP$-almost everywhere on $[0, T] \times \Omega$ and the condition *(iii)* in the statement is proved.

For the converse, let us fix $T > 0$, $\theta \in \mathbb{R}^n$, and $q \in L^2_{\mathcal{P}}([0, T], Ker\ D_1^*)$ such that $B_2^* p(s, q, \theta) = 0$, $P - a.s.$, for all $s \in [0, T]$. Since (3) is approximately null controllable we get $\theta = 0$ and

$$J(0, q(\cdot)) = 0.$$

Thus, $q(\cdot)$ is an optimal control for (SLQ) at $\theta = 0$ and the pathwise uniqueness yields

$$q(s) = 0 \ dsdP\text{-almost everywhere on } [0,T] \times \Omega.$$

The proof is now complete.

Remark 13 *The previous Proposition shows that, in order for (3) to be approximately -controllable, (SLQ) must be pathwise uniquely solvable at 0.*

The backward stochastic differential equation (9) may be interpreted as the following forward differential equation

$$\begin{cases} dp(t) = [-(A^* + C^*F)p(t) - C^*q(t)]\,dt + (Fp(t) + q(t))dW(t); \\ p(0) = \theta \in \mathbb{R}^n. \end{cases} \tag{10}$$

Therefore, for all $\theta \in \mathbb{R}^n$, all linear subspace $V \subset \mathbb{R}^n$, and all $q \in L^2_{\mathcal{P}}([0,T], V)$ there exists an unique predictable solution $p(\cdot, q, \theta)$ of (10) with continuous trajectories such that $E\left[\sup_{s\in[0,T]} |p(s,q,\theta)|^2\right] < \infty$. The approximate controllability conditions given by Proposition 9 become

Proposition 14 *The equation (3) is approximately-controllable if and only if, for all $T > 0$, all $\theta \in \mathbb{R}^n$, and all $q \in L^2_{\mathcal{P}}([0,T], Ker\ D_1^*)$ such that $B_2^*p(s,q,\theta) = 0$, $P-a.s.$, for all $s \in [0,T]$, it holds $q(s) = 0$, $dsdP$-almost everywhere on $[0,T] \times \Omega$ and $\theta = 0$.*

The equation (3) is approximately null controllable if and only if, for all $T > 0$, all $\theta \in \mathbb{R}^n$ and all $q \in L^2_{\mathcal{P}}([0,T], Ker\ D_1^*)$ such that $B_2^*p(s,q,\theta) = 0$, $P-a.s.$, for all $s \in [0,T]$, it holds $\theta = 0$.

3 Conditional local in time viability

In order to give a computable criterion for approximate controllability, the authors of [4] have introduced the notion of "local in time viability". Motivated by Proposition 14 and by the approach in [4], we extend the notion of "local in time viability" to "conditional local in time viability".

Definition 15 *Let U, $V \subset \mathbb{R}^n$ be two linear subspaces of $\mathbb{R}^n$. The family of all $\theta \in V$ for which there exists a $T > 0$ and $q \in L^2_{\mathcal{P}}([0,T], U)$ such that $p(s,q,\theta) \in V$, $P-a.s.$, for all $s \in [0,T]$ is called the viability kernel of V conditioned to U with respect to (10) (we denote this set by $Viab(V/U)$).*

Moreover, we say that V is local in time viable conditioned to U with respect to (10) if $Viab(V/U) = V$.

If U and V are two linear subspaces of $\mathbb{R}^n$, we denote by $\Pi_{U^\perp}$ (respectively $\Pi_{V^\perp}$) the orthogonal projections on $U^\perp$ (respectively $V^\perp$). For all $N \geq 1$, let us consider the following Riccati equations with values in $\mathcal{S}^n$ (the family of symmetric, non-negative matrix of $n \times n$ type):

$$\begin{cases} P_N'(s) = -P_N(s)(A^* + C^*F) - (A + F^*C)P_N(s) + F^*P_N(s)F \\ \qquad\qquad - (F^*P_N(s) - P_N(s)C^*)(I + N\Pi_{U^\perp} + P_N(s))^{-1}(P_N(s)F - CP_N(s)) \\ \qquad\qquad + N\Pi_{V^\perp}, \\ P_N(T) = 0; \end{cases} \tag{11}$$

Ito's formula applied to $\langle P_N(T-t)p(t), p(t)\rangle$ yields

$$E\left[\langle P_N(T-t)p(t), p(t)\rangle\right] = E\left[\int_t^T \left(N(|\Pi_{V^\perp}p(s)|^2 + |\Pi_{U^\perp}q(s)|^2) + |q(s)|^2\right) ds\right] \tag{12}$$

$$-E\left[\int_t^T \left|[f_N(P_N(s))]^{\frac{1}{2}}\left[[f_N(P_N(s))]^{-1}(P_N(s)F - CP_N(s))p(s) - q(s)\right]\right|^2 ds\right]$$

where $f_N(P_N(s)) = I + N\Pi_{U^\perp} + P_N(T-s)$.

Since $I + N\Pi_{U^\perp} \gg 0$ (that is $I + N\Pi_{U^\perp}$ is positive) and $N\Pi_{V^\perp} \geq 0$ (that is $N\Pi_{V^\perp}$ is nonnegative), the Riccati equation (11) admits an unique solution with values in $\mathcal{S}^n$ (see [10], condition (4.23) and Theorem 7.2).

Proposition 16 *The viability kernel of V conditioned to U with respect to (10) has the following representation:*

$$Viab(V|U) = \left\{\theta \in V : \ \exists T > 0 \ s.t.. \ \lim_{N\to\infty} \langle P_N(T)\theta, \theta\rangle < \infty\right\}.$$

PROOF. Let us consider $\theta \in Viab(V|U)$. Then, there exist $T > 0$ and $q \in L^2_{\mathcal{P}}([0,T], U)$ such that $p(s, q, \theta) \in V$, $P - a.s.$, for all $s \in [0,T]$. We recall that (12) holds true and get $E\left[\langle P_N(T)\theta, \theta\rangle\right] \leq E\int_0^T |q(s)|^2 ds$.

For the converse, we may choose, for all N, the optimal control process $\overline{q}_N \in L^2_{\mathcal{P}}([0,T], \mathbb{R}^n)$ and, again by (12), we have

$$\begin{aligned} \infty &> E\left[\langle P_N(T)\theta, \theta\rangle\right] \\ &= E\left[\int_0^T \left(N(|\Pi_{V^\perp}p(s, \overline{q}_N, \theta)|^2 + |\Pi_{U^\perp}\overline{q}_N(s)|^2) + |\overline{q}_N(s)|^2\right) ds\right]. \end{aligned}$$

The sequence $(\overline{q}_N)_{N\geq 0}$ is bounded in $L^2_{\mathcal{P}}([0,T], \mathbb{R}^n)$, and there exists a suitable subsequence (still denoted by $(\overline{q}_N)_{N\geq 0}$) such that $\overline{q}_N$ converges to $\overline{q}$ weakly in

$L^2_{\mathcal{P}}([0,T],\mathbb{R}^n)$. Since (10) is affine in q, we get the convergence $p(s,\overline{q}_N,\theta) \to p(s,\overline{q},\theta)$. Therefore, we have $\Pi_{V^\perp} p(s,\overline{q},\theta) = 0 \ P-a.s.$, for all $s \in [0,T]$. On the other hand, it can be easily seen that

$$E\left[\int_0^T N|\Pi_{U^\perp}\overline{q}_N(s)|^2 ds\right] \leq \lim_N \langle P_N(T)\theta, \theta\rangle ,$$

which leads to $E\left[\int_0^T |\Pi_{U^\perp}\overline{q}_N(s)|^2) ds\right] \to 0$ when $N \to \infty$. Let us notice the fact that

$$\begin{aligned}
E\left[\int_0^T |\Pi_{U^\perp}\overline{q}(s)|^2 ds\right] &= E\left[\int_0^T \langle \Pi_{U^\perp}\overline{q}(s), \overline{q}(s)\rangle\right] \\
&= \lim_{N\to\infty} E\left[\int_0^T \langle \Pi_{U^\perp}\overline{q}(s), \overline{q}_N(s)\rangle\right] \\
&\leq \lim_{N\to\infty} E\left[\int_0^T |\Pi_{U^\perp}\overline{q}_N(s)|^2\right]^{\frac{1}{2}} E\left[\int_t^T |\overline{q}(s)|^2\right]^{\frac{1}{2}} \\
&= 0
\end{aligned}$$

We deduce that $\overline{q} \in L^2_{\mathcal{P}}([0,T],U)$. The proof of our Proposition is now complete.

As in [4], one can show that

Proposition 17 *The viability kernel of the linear subspace $V \subset \mathbb{R}^n$ conditioned to the linear subspace $U \subset \mathbb{R}^n$ with respect to (10) is conditional locally in time viable. In particular, the conditional viability kernel is locally in time viable.*

PROOF. Let us consider $\theta \in Viab(V|U)$. Then, there exist $T > 0$ and $q \in L^2_{\mathcal{P}}([0,T],U)$ such that $p(s,q,\theta) \in V$, $P-a.s.$, for all $s \in [0,T]$. Therefore, using (12), we have

$$E\left[\langle P_N(T-s)p(s,q,\theta), p(s,q,\theta)\rangle\right] \leq E\left[\int_s^T |q(r)|^2 dr\right].$$

Thus, by a monotone convergence argument and Proposition 16, we infer that $p(s,q,\theta) \in Viab(V|U) \ P-a.s.$

4 The main result

We are now able to prove our main result:

Theorem 18 *We have equivalence between the following assertions:*

(1) The equation (3) is approximately-controllable.

(2) The equation (3) is approximately null controllable.

(3) The viability kernel of $Ker\ B_2^*$ conditioned to $Ker\ D_1^*$ is trivial.

PROOF. We only have to prove *(3)$\Longrightarrow$(1)*. In order to establish the result, suppose that $Viab(Ker\ B_2^*|Ker\ D_1^*)$ is trivial and let $q \in L^2_{\mathcal{P}}([0,T], Ker D_1^*)$ such that $p(s,q,\theta) \in Ker\ B_2^*$, $P-a.s.$, for all $s \in [0,T]$. We get $\theta = 0$, and $p(s,q,\theta) \in Viab(Ker\ B_2^*|Ker\ D_1^*)$. Therefore, $p(s,q,\theta) = 0$, $P-a.s.$, for all $s \in [0,T]$. Recall that $p(\cdot,q,\theta)$ is the solution of (10) to conclude $q(s) = 0$, $dsdP$-almost everywhere on $[0,T] \times \Omega$.

The following Theorem gives a method to obtain conditional local in time viability using conditional strict invariance, thus providing an algebraic criterion for approximate controllability.

Theorem 19 *The linear subspace $V \subset \mathbb{R}^n$ is local in time viable conditioned to the linear subspace $U \subset \mathbb{R}^n$ with respect to (10) if and only if V is $(A^*;C^*)$-strictly invariant conditioned to (F,U).*

PROOF. Let us first suppose that V is $(A^*;C^*)$-strictly invariant conditioned to (F,U). We wish to prove that V is local in time viable conditioned to U. In order to prove this, it suffices to notice that there exists a linear operator $K \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ such that $KV \subset V$, $(K-F)V \subset U$ and $(A^* + C^*K)V \subset V$. Indeed, for any $\theta \in V$, we consider the linear stochastic differential equation

$$\begin{cases} d\overline{p}(t) = -(A^* + C^*K)p(t)dt + Kp(t)dW(t), \\ \overline{p}(0) = \theta. \end{cases}$$

Obviously, the solution of this equation is in V. Moreover, if we set $q(t) = (K-F)\overline{p}(t) \in U$, we notice that $p(t,q(t),\theta) = \overline{p}(t) \in V$ for all $t > 0$.

For the converse let us fix $\theta \in V$. We suppose that $p(s,q,\theta) \in V$, $P-a.s.$, for all $s \in [0,T]$, for some $T > 0$ and $q \in L^2_{\mathcal{P}}([0,T], U)$. We multiply (10) with $(I - \Pi_V)$ to obtain

$$\begin{cases} d(I-\Pi_V)p(t) = (I-\Pi_V)\left[-(A^* + C^*F)p(t) - C^*q(t)\right]dt \\ \qquad\qquad +(I-\Pi_V)(Fp(t) + q(t))dW(t), \\ p(0) = \theta. \end{cases}$$

Since we have supposed that $p(s,q,\theta) \in V$, $P-a.s.$, the quadratic variation of $(I-\Pi_V)p$ is zero (i.e. $Fp(t)+q(t) \in V$, $dtdP$-almost everywhere on $[0,T]\times\Omega$). Moreover, we have $(I-\Pi_V)\left[-(A^*+C^*F)p(t)-C^*q(t)\right]=0$, $dtdP$-almost everywhere on $[0,T]\times\Omega$.

At this point, we consider the linear subspace $W \subset V$

$$W=\left\{\theta \in V \text{ s.t. } \exists \alpha \in V:\ \alpha - F\theta \in U,\ A^*\theta + C^*\alpha \in V\right\}.$$

It is straightforward that $p(t,q,\theta) \in W$, $dtdP$-almost everywhere on $[0,T]\times\Omega$. We use the continuity of the trajectories of p to finally get $\theta \in W$. This implies that $V=W$ (i.e. V is $(A^*;C^*)$-strictly invariant conditioned to (F,U)).

Since the viability kernel of $Ker\ B_2^*$ conditioned to $Ker\ D_1^*$ is locally in time viable conditioned to $Ker\ D_1^*$, we deduce that $Viab(KerB_2^*|KerD_1^*)$ is the largest space which is $(A^*;C^*)$-strictly invariant conditioned to $(F,KerD_1^*)$ for some F solution for the equation $D_1^*F+B_1^*=0$. Thus, we can state our final result

Theorem 20 *We have equivalence between the following assertions:*

(1) The equation (3) is approximately-controllable.

(2) The equation (3) is approximately null controllable.

(3) The largest linear subspace of $KerB_2^*$ which is $(A^*;C^*)$-strictly invariant conditioned to $(F,KerD_1^*)$ is the trivial subspace $\{0\}$.

The latter assertion is easily computable as seen in the Remark 7.

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Approximate Controllability for Linear Stochastic Differential Equations in Infinite Dimensions

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Abstract The objective of the paper is to investigate the approximate controllability property of a linear stochastic control system with values in a separable real Hilbert space. In a first step we prove the existence and uniqueness for the solution of the dual linear backward stochastic differential equation. This equation has the particularity that in addition to an unbounded operator acting on the Y-component of the solution there is still another one acting on the Z-component. With the help of this dual equation we then deduce the duality between approximate controllability and observability. Finally, under the assumption that the unbounded operator acting on the state process of the forward equation is an infinitesimal generator of an exponentially stable semigroup, we show that the generalized Hautus test provides a necessary condition for the approximate controllability. The paper generalizes former results by Buckdahn, Quincampoix and Tessitore (2006) and Goreac (2007) from the finite dimensional to the infinite dimensional case.

1 Preliminaries

This paper is concerned with the study of approximate controllability of an infinite dimensional stochastic equation with multiplicative noise

$$\begin{cases} dX_t^{x,u} = (AX_t^{x,u} + Bu_t)\,dt + CX_t^{x,u}dW_t, \\ X_0 = x \in H, \end{cases} \tag{1}$$

where u is a U-valued stochastic control process, and the state space H as well as the control state space U are separable real Hilbert spaces. We say that the above equation enjoys the approximate controllability property if, for any initial data $x \in H$, and all finite time horizon $T > 0$, one can find a control process u which keeps the solution $X_T^{x,u}$ arbitrarily close to a given square integrable final condition.

For deterministic control systems with finite dimensional state space $\mathbb{C}^n$, controllability is completely characterized by the well-known Kalman condition. Often, it is convenient to study the observability of the adjoint system rather than the controllability of the initial system. Indeed, whenever dealing with a deterministic control system

$$\begin{cases} dX_t^{x,u} = (AX_t^{x,u} + Bu_t)\,dt, \\ X_0 = x \in \mathbb{C}^n, \end{cases} \tag{2}$$

controllability is equivalent to the observability of the dual system

$$\begin{cases} dY_t^y = -A^* Y_t^y dt,\ O_t^y = B^* Y_T^y, \\ Y_0^y = y. \end{cases} \tag{3}$$

A very powerful tool for this approach is the Hautus test. According to this test, observability of (3) (and, thus, controllability for (2)) is equivalent to

$$rank \begin{bmatrix} sI - A^* \\ B^* \end{bmatrix} = n, \text{ for all } s \in \mathbb{C}.$$

In the case of separable Hilbert state space, whenever A generates an exponentially stable semigroup, Russell and Weiss [20] have obtained a necessary condition for observability which generalizes the Hautus criterion. They have also conjectured that this condition is even sufficient. Jacob and Zwart [14] proved that the above conjecture holds true for the class of diagonal systems satisfying the strong stability condition whenever the output space is finite dimensional. Similar arguments allow to obtain in [13] a characterization of approximate controllability of a deterministic controlled system with 1-dimensional input.

In the stochastic framework, Kalman-type characterizations of approximate controllability have been obtained, for the finite-dimensional case, by Buckdahn, Quincampoix and Tessitore [3] when the noise term is not controlled, and by Goreac [11] when the control is allowed to act on the noise. The method they use relies on the duality between approximate controllability and approximate observability for the dual equation. Riccati algebraic arguments allow to obtain in [3] and [11] an invariance criterion for the approximate controllability of the initial system.

In the case of controlled stochastic systems with infinite-dimensional state space, we cite Barbu, Răşcanu, Tessitore [1], Fernandez-Cara, Garrido-Atienza, Real [8], and Sirbu, Tessitore [21]. In [21], the authors characterize the property of (null) controllability with the help of singular Riccati equations. They also provide a Riccati characterization using the duality approach.

In this paper, we prove the duality between approximate controllability for the forward system and some approximate observability for the dual system, and we use this approach to show that the generalized Hautus test is a necessary condition for approximate controllability whenever A is the generator of an exponentially stable semigroup.

The paper is organized as follows: In the first section we introduce the standard notations and assumptions which will be used in what follows. After, in the second section, we investigate the existence and the uniqueness of the mild solution of the following backward stochastic differential equation which is associated as dual equation to the controlled system (1):

$$\begin{cases} dY_t = -\left(A^* Y_t + C^* Z_t\right) dt + Z_t dW_t, \\ \quad Y_T = \xi \in L^2\left(\Omega, \mathcal{F}_T, P; H\right). \end{cases}$$

We emphasize that the drift term in our dual backward equation contains not only the unbounded operator A^* acting on Y but also the unbounded operator C^* that acts on Z. To overcome the difficulties related with, we make a joint dissipativity hypothesis which corresponds, in the case of general heat equations, to the usual joint ellipticity condition. Under these minimal assumptions we are able to prove the existence and the uniqueness. Moreover, we provide a duality result between approximate controllability for the forward equation and the approximate observability of the dual system. The third section proves that, whenever A generates an exponentially stable semigroup, the Russell and Weiss generalization of the Hautus test is a necessary condition for approximate controllability of stochastic systems. Finally, we discuss as example the general heat equation.

2 Introduction

Let us begin by introducing some basic notations and standard assumptions. The spaces $(H, \langle\cdot,\cdot\rangle_H)$, $(U, \langle\cdot,\cdot\rangle_U)$, $(\Xi, \langle\cdot,\cdot\rangle_\Xi)$ are separable real Hilbert spaces. We let $\mathcal{L}(\Xi, H)$ denote the space of all bounded H-valued linear operators on Ξ, and $L_2(\Xi, H)$ be the subspace of Hilbert-Schmidt operators. Both spaces are endowed with the usual norms. Moreover, we consider a linear dissipative operator $A : D(A) \subset H \longrightarrow H$ which generates a C_0-semigroup of linear operators $\left(e^{tA}\right)_{t\geq 0}$, a linear bounded operator $B \in \mathcal{L}(U, H)$ and a linear operator $C : H \longrightarrow \mathcal{L}(\Xi, H)$ such that, for all $t > 0$,

$$a)\ e^{tA} C \in \mathcal{L}\left(H; L_2(\Xi, H)\right),$$
$$b)\ \left|e^{tA} C\right|_{\mathcal{L}(H; L_2(\Xi,H))} \leq L t^{-\gamma},$$

for some constants $\gamma \in \left[0, \frac{1}{2}\right)$ and $L > 0$.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space endowed with a filtration $(\mathcal{F}_t)_{t\geq 0}$ which is supposed to satisfy the usual assumptions of completeness and right-continuity. We denote by W a cylindrical $(\mathcal{F}_t)-$Wiener process that takes its values in Ξ. Finally, we let $\mathcal{U}$ denote the space of all $(\mathcal{F}_t)$ −progressively measurable processes $u : \mathbb{R}_+ \times \Omega \longrightarrow U$ such that

$$E\left[\int_0^T |u_t|^2 \, dt\right] < \infty, \text{ for all } T > 0.$$

.

The aim of this paper is to give an easy and verifiable criterion for approximate controllability for the following linear stochastic differential equation

$$\begin{cases} dX_t^{x,u} = \left(AX_t^{x,u} + Bu_t\right) dt + CX_t^{x,u} dW_t, \ t \geq 0. \\ X_0 = x \in H. \end{cases} \tag{4}$$

Given an admissible control process $u \in \mathcal{U}$, an $(\mathcal{F}_t)$-progressively measurable process $X^{x,u}$ with

$$E\left[\sup_{s\in[0,T]} |X_s^{x,u}|^2\right] < \infty, \text{ for all } T > 0,$$

is a mild solution of (4) if, for all $t > 0$,

$$X_t = e^{tA}x + \int_0^t e^{sA} Bu_s ds + \int_0^t e^{sA} CX_s dW_s, \tag{5}$$

P-a.s. Under the standard assumptions given above, there exists a unique mild solution of (4). For further results on mild solutions, the reader is referred to Da Prato, Zabczyk [5], and Fuhrman, Tessitore [9].

3 The dual equation

Let us now consider the following backward stochastic differential equation

$$\begin{cases} dY_t = -\left(A^* Y_t + C^* Z_t\right) dt + Z_t dW_t, \\ \qquad Y_T = \xi \in L^2\left(\Omega, \mathcal{F}_T, P; H\right). \end{cases} \tag{6}$$

Since $C : H \longrightarrow \mathcal{L}(\Xi, H)$, also $Ce^{tA} : H \longrightarrow \mathcal{L}(\Xi, H)$, for all $t \geq 0$. Let us assume that, for all $t > 0$, all the values of Ce^{tA} are in $L_2(\Xi; H)$,

$$Ce^{tA} : H \longrightarrow L_2(\Xi; H).$$

Then, of course, the linear operator $\left(Ce^{tA}\right)^*$ maps $L_2(\Xi; H)$ into H and we can introduce the notion of a mild solution for equation (6). A mild solution of (6) is a couple (Y, Z) of progressively measurable processes with values in H, respectively $L_2(\Xi, H)$, such that

$$\begin{cases} (Y, Z) \in C\left([0, T]; L^2(\Omega; H)\right) \times L^2\left([0, T] \times \Omega; L_2(\Xi; H)\right), \\ \sup_{t\in[0,T]} E\left[|Y_t|^2\right] + E\left[\int_0^T |Z_t|^2 \, dt\right] < \infty, \\ \int_0^T \left|\left(Ce^{(s-t)A}\right)^* Z_s\right| ds < \infty, \ P - a.s., \\ \\ Y_t = e^{(T-t)A^*}\xi + \int_t^T \left(Ce^{(s-t)A}\right)^* Z_s ds - \int_t^T e^{(s-t)A^*} Z_s dW_s, \ t \in [0, T]. \end{cases}$$

If C is a bounded linear operator, then it has been shown in Confortola [[4], Th. 2.2] that (6) admits a unique mild solution. Let us suppose that

(A1) *The operator C may be written as sum of two linear operators C_1, C_2*

$$C = C_1 + C_2,$$

satisfying the following properties:

1) C_2 is a bounded operator from H to $L_2(\Xi; H)$,

2) for all $t > 0$, $C_1 e^{tA} \in L(H; L_2(\Xi; H))$. Moreover, we suppose that there exist some $\gamma \in \left[0, \frac{1}{2}\right)$ and some positive constant $L > 0$ such that

$$\left|C_1 e^{tA}\right|_{\mathcal{L}(H;L_2(\Xi;H))} \leq Lt^{-\gamma},$$

for all $t > 0$.

3) There exists some constant $a > \frac{1}{2}$ such that

$$A + a\left(C_1 e^{\delta A}\right)^* \left(C_1 e^{\delta A}\right) \text{ is dissipative,}$$

for some sequence $\delta \searrow 0$.

If C_1 is different of zero, we shall also assume that

(A2) *$-A^2$ is dissipative.*

Remark 1 If A is a self-adjoint, dissipative operator which generates a contraction semigroup, then (A.2) is obviously satisfied.

Moreover, if we suppose that C_1 takes its values in $L_2(\Xi; H)$, then we may replace (A1) 3) by

3') there exists some constant $a > \frac{1}{2}$ such that

$$A + aC_1^* C_1 \text{ is dissipative.}$$

Indeed, in this case $e^{2\delta A}$ is a bounded operator which commutes with the self-adjoint positive operator $-A$ and also with its square root $\sqrt{-A}$. Thus, for all $x \in D(A)$,

$$\left\langle e^{2\delta A}(-A)x, x\right\rangle = \left|e^{\delta A}\sqrt{-A}x\right|^2 \leq \left|\sqrt{-A}x\right|^2 = \langle(-A)x, x\rangle.$$

It follows that $A - e^{\delta A} A e^{\delta A^*}$ is dissipative. Therefore, also $A + ae^{\delta A^*} C_1^* C_1 e^{\delta A}$ is dissipative.

We now can state the main result of this section.

Theorem 1 *Under the assumptions (A1) and (A2), there exists a unique mild solution of the backward linear stochastic differential equation (6). Moreover, this solution satisfies*

$$\sup_{t\in[0,T]} E\left[|Y_t|^2\right] + E\left[\int_0^T |Z_s|^2\, ds\right] \leq kE\left[|\xi|^2\right], \tag{7}$$

where $k > 0$ is some constant that doesn't depend on the particular choice of ξ but only on the operators A, C and the time horizon T.

Remark 2 1. The existence and uniqueness of the solution for equation (6) has been studied by Tessitore [22] for the case in which A generates an analytic semigroup of contractions of negative type; the Brownian motion was supposed to be finite-dimensional. His main assumption, the joint dissipativity condition, was justified by its necessity for the "well-posedness" and coercivity of the forward system. The approach is fundamentally different from ours and relies on duality methods. However, let us point out that the author obtains, for his analytic case, stronger space regularity properties for the solution of the BSDE.

2. Ma, Yong [17] treated a particular linear, degenerate BSPDE. Their method relies on a parabolicity assumption and a priori estimates that allowed the authors to get the well-posedness of the problem, the existence, the uniqueness as well as regularity properties. Later the same technique was used by Hu, Ma, Yong [12] for further extensions.

Proof (of Theorem 1). We begin by proving the existence: The main difficulty to prove the existence and the uniqueness for a BSDE in infinite dimensions with unbounded linear operators consists in the fact that Itô's formula can't be applied directly to this equation because it is defined only in the mild sense. To overcome this difficulty, we have to reduce the problem with the help of two different approximations to BSDEs that allow the application of Itô's formula. We first approximate our original BSDE by the following one:

$$\begin{cases} dY_t^\delta = -A^* Y_t^\delta dt - (C_1 e^{\delta A})^* Z_t^\delta dt - C_2^* Z_t^\delta dt + Z_t^\delta dW_t, \\ Y_T^\delta = \xi \in L^2(\Omega, \mathcal{F}_T, P; H) \end{cases} \tag{8}$$

For this approximating equation we know that, due to the results of Confortola [4], there exists a unique mild solution (Y^δ, Z^δ) for every $\delta > 0$.

In a first step we prove that

Step 1. There is a positive constant k independent of $\delta > 0$ and ξ such that

$$\sup_{t\in[0,T]} E\left[\left|Y_t^\delta\right|^2\right] + E\left[\int_0^T \left|Z_s^\delta\right|^2 ds\right] \le kE\left|\xi\right|^2. \tag{9}$$

Indeed, we introduce the Yosida approximation of the dissipative operator A^*, $A_n^* = n(nI - A^*)^{-1}A^* = J_n^* A^*$, and we consider the following approximating BSDE:

$$\begin{cases} dY_t^{n,\delta} = -\, A_n^* Y_t^{n,\delta} dt - J_n^* \left(C_1 e^{\delta A}\right)^* Z_t^{n,\delta} dt - C_2^* Z_t^{n,\delta} dt + Z_t^{n,\delta} dW_t, \\ Y_T^{n,\delta} = \xi \in L^2(\Omega, \mathcal{F}_T, P; H). \end{cases}$$

It is well known that the above equation admits a unique solution $\left(Y^{n,\delta}, Z^{n,\delta}\right)$. Let $1 < \alpha < 2a$ and $\beta > 0$ be such that $\frac{1}{\alpha} + \frac{1}{\beta} < 1$. Then, by applying Itô's formula to $\left|Y^{n,\delta}\right|^2$ we obtain

$$
\begin{aligned}
E\left|\xi\right|^2 &= E\left[\left|Y_t^{n,\delta}\right|^2\right] - 2E\left[\int_t^T \left\langle A_n^* Y_s^{n,\delta}, Y_s^{n,\delta}\right\rangle\right] \\
&- 2E\left[\int_t^T \left\langle J_n^* \left(C_1 e^{\delta A}\right)^* Z_s^{n,\delta}, Y_s^{n,\delta}\right\rangle\right] \\
&- 2E\left[\int_t^T \left\langle C_2^* Z_s^{n,\delta}, Y_s^{n,\delta}\right\rangle\right] + E\left[\int_t^T \left|Z_s^{n,\delta}\right|^2 ds\right] \\
&\geq E\left[\left|Y_t^{n,\delta}\right|^2\right] + \left(1 - \frac{1}{\alpha} - \frac{1}{\beta}\right) E\left[\int_t^T \left|Z_s^{n,\delta}\right|^2 ds\right] \\
&- 2E\left[\int_t^T \left\langle\left(A_n^* + \frac{\alpha}{2} J_n^* \left(C_1 e^{\delta A}\right)^* \left(C_1 e^{\delta A}\right) J_n\right) Y_s^{n,\delta}, Y_s^{n,\delta}\right\rangle\right] \\
&- \beta |C_2^*|^2 E\left[\int_t^T \left|Y_s^{n,\delta}\right|^2 ds\right], \qquad (10)
\end{aligned}
$$

On the other hand, with the help of assumption (A.2) we can prove that

$$
\begin{aligned}
&A_n^* + \frac{\alpha}{2} J_n^* \left(C_1 e^{\delta A}\right)^* \left(C_1 e^{\delta A}\right) J_n \\
&= -n^{-1} A_n^* A_n + J_n^* \left(A^* + \frac{\alpha}{2} \left(C_1 e^{\delta A}\right)^* \left(C_1 e^{\delta A}\right)\right) J_n
\end{aligned}
$$

is a dissipative operator. It then follows from (10) that

$$
\begin{aligned}
&E\left[\left|Y_t^{n,\delta}\right|^2\right] + \left(1 - \frac{1}{\alpha} - \frac{1}{\beta}\right) E\left[\int_t^T \left|Z_s^{n,\delta}\right|^2 ds\right] \\
&\leq E\left|\xi\right|^2 + \beta |C_2|^2 E\left[\int_t^T \left|Y_s^{n,\delta}\right|^2 ds\right],
\end{aligned}
$$

and Gronwall's inequality yields

$$
\sup_{t \in [0,T]} E\left[\left|Y_t^{n,\delta}\right|^2\right] + E\left[\int_0^T \left|Z_s^{n,\delta}\right|^2 ds\right] \leq kE\left|\xi\right|^2, \qquad (11)
$$

Notice that the constant k here is independent of $n \geq 1, \delta > 0$ and of ξ; it denotes a generic constant whose value can change from line to line. From the above estimate we can conclude that there is a subsequence, still denoted $\left(Y^{n,\delta}, Z^{n,\delta}\right)_n$, such that $Y^{n,\delta} \to Y^\delta$ weakly * in $L^\infty\left([0,T]; L^2(\Omega; H)\right)$ and $Z^{n,\delta} \to Z^\delta$ weakly in $L^2\left(\Omega \times [0,T]; L_2(\Xi; H)\right)$. It can be easily proved the limit (Y^δ, Z^δ) is the unique mild solution of (8). This allows to consider for Y^δ its version in $C\left([0,T]; L^2(\Omega; H)\right)$. Finally, from Mazur's theorem we obtain that (Y^δ, Z^δ) satisfies the estimate announced in step 1.

In preparation of the next step we observe that, since $(Y^\delta, Z^\delta)_{\delta>0}$ is bounded in $L^\infty\left([0,T];L^2(\Omega;H)\right) \times L^2\left(\Omega\times[0,T];L_2(\Xi;H)\right)$, we get the existence of some subsequence, again denoted by $(Y^\delta, Z^\delta)_{\delta>0}$, such that $Y^\delta \to Y$ weak * in $L^\infty\left([0,T];L^2(\Omega;H)\right)$ and $Z^\delta \to Z$ weakly in $L^2\left(\Omega\times[0,T];L_2(\Xi;H)\right)$, as $\delta\to 0$.

We want to prove that the couple (Y,Z) obtained above is a mild solution of our BSDE:

$$
\begin{aligned}
Y_t &= e^{(T-t)A^*}\xi + \int_t^T \left(C_1 e^{(s-t)A}\right)^* Z_s ds \\
&\quad + \int_t^T e^{(s-t)A^*} C_2^* Z_s ds - \int_t^T e^{(s-t)A^*} Z_s dW_s.
\end{aligned} \tag{12}
$$

For this we notice that, since (Y^δ, Z^δ) is a mild solution of (8), we have

$$
\begin{aligned}
Y_t^\delta &= e^{(T-t)A^*}\xi + \int_t^T e^{(s-t)A^*}\left(C_1 e^{\delta A}\right)^* Z_s^\delta ds \\
&\quad + \int_t^T e^{(s-t)A^*} C_2^* Z_s^\delta ds - \int_t^T e^{(s-t)A^*} Z_s^\delta dW_s
\end{aligned} \tag{13}
$$

and we show the following:

<u>Step 2</u> The process

$$
M_t^{1,\delta} = \int_t^T e^{(s-t)A^*}\left(C_1 e^{\delta A}\right)^* Z_s^\delta ds,\ t\in[0,T],
$$

belongs to $L^\infty\left([0,T];L^2(\Omega;H)\right)$ and converges weakly * in $L^\infty\left([0,T];L^2(\Omega;H)\right)$ to $M^1 = \left(\int_t^T \left(C_1 e^{(s-t)A}\right)^* Z_s ds\right)_{t\in[0,T]}$.

Indeed, by using that

$$
e^{\delta' A^*}\left(C_1 e^{\delta A}\right)^* = \left(C_1 e^{\left(\delta+\delta'\right)A}\right)^*,
$$

for all $\delta, \delta' > 0$, we have

$$
\begin{aligned}
& E\left[\left|\int_t^T e^{(s-t)A^*}\left(C_1 e^{\delta A}\right)^* Z_s^\delta ds\right|^2\right] \\
&\le E\left[\left(\int_t^T \left|e^{\delta A^*}\left(C_1 e^{(s-t)A}\right)^* Z_s^\delta\right| ds\right)^2\right] \\
&\le kE\left[\int_t^T (s-t)^{-2\gamma} ds \int_t^T \left|Z_s^\delta\right|^2 ds\right] \\
&\le kE\left|\xi\right|^2,
\end{aligned}
$$

which implies that $\{M^{1,\delta}, \delta > 0\} \subset L^\infty\left([0,T]; L^2(\Omega; H)\right)$ is bounded. Moreover, for all $\phi \in L^2(\Omega; H)$ and $t \in [0,T]$,

$$
\begin{aligned}
E\left[\left\langle M_t^{1,\delta}, \phi\right\rangle\right] &= E\left[\int_t^T \left\langle \left(C_1 e^{(s-t)A}\right)^* Z_s^\delta, \left(e^{\delta A^*} - I\right)\phi\right\rangle ds\right] \\
&\quad + E\left[\int_t^T \left\langle \left(C_1 e^{(s-t)A}\right)^* Z_s^\delta, \phi\right\rangle ds\right] =: I_1^\delta + I_2^\delta, \qquad (14)
\end{aligned}
$$

where

$$
\begin{aligned}
I_1^\delta &= E\left[\left|\int_t^T \left\langle \left(C_1 e^{(s-t)A}\right)^* Z_s^\delta, \left(e^{\delta A^*} - I\right)\phi\right\rangle ds\right|\right] \\
&\leq E\left[\left|\left(e^{\delta A^*} - I\right)\phi\right| \int_t^T \left|\left(C_1 e^{(s-t)A}\right)^* Z_s^\delta\right| ds\right] \\
&\leq \left(E\left[\int_t^T (s-t)^{-2\gamma}\, ds \int_t^T \left|Z_s^\delta\right|^2 ds\right]\right)^{\frac{1}{2}} \left(E\left[\left|\left(e^{\delta A^*} - I\right)\phi\right|^2\right]\right)^{\frac{1}{2}} \\
&\leq k\left(E\left|\xi^2\right|\right)^{\frac{1}{2}} \left(E\left[\left|\left(e^{\delta A^*} - I\right)\phi\right|^2\right]\right)^{\frac{1}{2}}.
\end{aligned}
$$

Consequently, due to the dominated convergence theorem,

$$
I_1^\delta \to 0 \text{ as } \delta \to 0.
$$

For the second term we have

$$
I_2^\delta = E\left[\int_t^T \left\langle \left(C_1 e^{(s-t)A}\right)^* Z_s^\delta, \phi\right\rangle ds\right] = E\left[\int_t^T \left\langle Z_s^\delta, \left(C_1 e^{(s-t)A}\right)\phi\right\rangle ds\right],
$$

and since $\left(C_1 e^{(s-t)A}\right)\phi \in L^2\left(\Omega \times [0,T]; L_2(\Xi; H)\right)$, it follows from the weak convergence of Z^δ to Z that

$$
I_2^\delta = E\left[\int_t^T \left\langle \left(C_1 e^{(s-t)A}\right)^* Z_s^\delta, \phi\right\rangle ds\right] \to E\left[\int_t^T \left\langle \left(C_1 e^{(s-t)A}\right)^* Z_s, \phi\right\rangle ds\right],
$$

and from (14) we then get

$$
E\left[\left\langle M_t^{1,\delta}, \phi\right\rangle\right] \to E\left[\langle M_t^1, \phi\rangle\right] \text{ as } \delta \to 0.
$$

In order to prove that $M^{1,\delta}$ converges in the weak * topology on $L^\infty([0,T]; L^2(\Omega; H))$ to M^1, we consider $\Phi \in L^1\left([0,T]; L^2(\Omega; H)\right)$, and use the fact that, for all $t \in [0,T]$ for which $\Phi_t \in L^2(\Omega; H)$, the previous convergence holds with Φ_t at the place of ϕ. We then apply a dominated convergence argument and get the statement of step 2.

Step 3. The couple (Y,Z) is a solution of the BSDE

$$Y_t = e^{(T-t)A^*}\xi + \int_t^T \left(C_1 e^{(s-t)A}\right)^* Z_s ds + \int_t^T e^{(s-t)A^*} C_2^* Z_s ds - \int_t^T e^{(s-t)A^*} Z_s dW_s. \qquad (15)$$

Moreover,

$$\sup_{t\in[0,T]} E\left[|Y_t|^2\right] + E\left[\int_0^T |Z_s|^2 ds\right] \le kE\,|\xi|^2. \qquad (16)$$

To prove the above statement we write Y_t^δ, $t\in[0,T]$, as

$$Y_t^\delta = e^{(T-t)A^*}\xi + M_t^{1,\delta} + M_t^{2,\delta} + M_t^{3,\delta}.$$

While we have already studied the convergence of $M^{1,\delta}$ in the preceding step, it is an immediate consequence of the boundedness of the operator C_2 that $M_t^{2,\delta} = \int_t^T e^{(s-t)A^*} C_2^* Z_s^\delta ds$ converges weakly * in $L^\infty\left([0,T]; L^2(\Omega;H)\right)$ to $M_t^2 = \int_t^T e^{(s-t)A^*} C_2^* Z_s ds$.

For the noise term $M_t^{3,\delta} = \int_t^T e^{(s-t)A^*} Z_s^\delta dW_s$ we notice that since Z^δ converges weakly in $L^2\left(\Omega\times[0,T]; L_2(\Xi;H)\right)$ to Z, $e^{(\cdot-t)A^*}Z_\cdot^\delta$ also converges weakly to $e^{(\cdot-t)A^*}Z_\cdot$. We apply the martingale representation theorem to get that
$\int_t^T e^{(s-t)A^*} Z_s^\delta dW_s$ converges weakly in $L^2(\Omega;H)$ to $\int_t^T e^{(s-t)A^*} Z_s dW_s$. Using, as before, the dominated convergence, we get that

$$N_t^\delta = \int_t^T e^{(s-t)A^*} Z_s^\delta dW_s \text{ converges in the weak* topology on}$$
$$L^\infty\left([0,T]; L^2(\Omega;H)\right) \text{ to } N_t = \int_t^T e^{(s-t)A^*} Z_s dW_s.$$

We now pass to the $L^\infty\left([0,T]; L^2(\Omega;H)\right)$ weak * limit in the approximating mild equation (13). This yields the statement of step 3, with the only difference, that for the BSDE which has been got by a weak limit, we only know for the moment that this equation is satisfied $dtdP$-a.e. To obtain that the BSDE is satisfied by (Y,Z) for all time points of the interval $[0,T]$, P-a.s., we need the following auxiliary statement:

Lemma 1 *The process*
$\Phi_t = e^{(T-t)A^*}\xi + \left(C_1 e^{(r-t)A}\right)^* Z_r dr + \int_t^T e^{(r-t)A^*} C_2^* Z_r dr$
$- \int_t^T e^{(r-t)A^*} Z_r dW_r$, $t\in[0,T]$, *is mean-square continuous.*

Proof We return to the proof of our theorem. The proof of the lemma will be given afterwards.

The above result allows to conclude the proof of step 3. Indeed, the above lemma guarantees the existence of a version of the solution (Y,Z) in

$C\left([0,T];L^2(\Omega;H)\right)\times L^2(\Omega\times[0,T];L_2(\Xi;H))$. For this version we have (15) for all $t\in[0,T]$.

Let us prove now the uniqueness of the solution of our BSDE. In virtue of the linearity of the equation it suffices to prove the following:

Step 4. The only solution (Y,Z) of the BSDE

$$\begin{cases} dY_t = -A^*Y_t dt - C^*Z_t dt + Z_t dW_t, \\ Y_T = 0. \end{cases}$$

is the trivial one: $(Y,Z)=(0,0)$.

To prove this, we have to transform the BSDE into an equation which allows to apply Itô's formula. For this reason we put, for all $n\geq 1$ and $\delta>0$,

$$\widetilde{Y}_\cdot := J_n^* e^{\delta A^*} Y_\cdot,$$

and we observe that the such introduced process $\widetilde{Y}$ satisfies the following backward equation:

$$\begin{cases} d\widetilde{Y}_t = -A^*\widetilde{Y}_t dt - J_n^*\left(C_1 e^{\delta A}\right)^* Z_t dt - J_n^* e^{\delta A^*} C_2^* Z_t dt + J_n^* e^{\delta A^*} Z_t dW_t, \\ \widetilde{Y}_T = 0. \end{cases}$$

To this equation we can apply Itô's formula (Indeed, notice that $A^*\widetilde{Y}_\cdot = (J_n^* e^{\delta A^*}A^*)Y_\cdot$, where the operator $J_n^* e^{\delta A^*}A^*$ is bounded). This yields:

$$\begin{aligned}
0 &= E\left[\left|J_n^* e^{\delta A^*} Y_t\right|^2\right] - 2E\left[\int_t^T \left\langle A^*\widetilde{Y}_s, \widetilde{Y}_s\right\rangle ds\right] \\
&- 2E\left[\int_t^T \left\langle J_n^*\left(C_1 e^{\delta A}\right)^* Z_s, \widetilde{Y}_s\right\rangle ds\right] \\
&- 2E\left[\int_t^T \left\langle J_n^* e^{\delta A^*} C_2^* Z_s, \widetilde{Y}_s\right\rangle ds\right] + E\left[\int_t^T \left|J_n^* e^{\delta A^*} Z_s\right|^2 ds\right] \\
&\geq E\left[\left|J_n^* e^{\delta A^*} Y_t\right|^2\right] - 2E\left[\int_t^T \left\langle\left(A^* + \frac{\alpha}{2}J_n^*\left(C_1 e^{\delta A}\right)^*\left(C_1 e^{\delta A}\right)J_n\right)\widetilde{Y}_s, \widetilde{Y}_s\right\rangle ds\right] \\
&- \beta\left|C_2\right|^2 E\left[\int_t^T \left|Y_s\right|^2 ds\right] + E\left[\int_t^T \left|J_n^* e^{\delta A^*} Z_s\right|^2 ds\right] \\
&- \left(\frac{1}{\alpha}+\frac{1}{\beta}\right) E\left[\int_t^T \left|Z_s\right|^2 ds\right],
\end{aligned} \tag{17}$$

To be able to go ahead with the above estimate we need the dissipativity of the operator $A^* + \frac{\alpha}{2}J_n^*\left(C_1 e^{\delta A}\right)^*\left(C_1 e^{\delta A}\right)J_n$.

For this end we notice that

$$\left(nI - A^*\right)A^*\left(nI - A\right) - n^2 A^* = -nA^*A^* - nA^*A + A^*A^*A$$

and apply this relation to the operator $(nI-A)^{-1}$. To the relation we then apply $(nI-A^*)^{-1}$. So we obtain the following equality:

$$A^* - J_n^* A^* J_n = -n^{-1} J_n^* (A^*)^2 J_n - n^{-1} J_n^* A^* A J_n + n^{-2} J_n^* A^* A^* A J_n,$$

which proves that the operator $A^* - J_n^* A^* J_n$ is dissipative. It now follows easily that also the operator

$$\begin{aligned} & A^* + \frac{\alpha}{2} J_n^* \left(C_1 e^{\delta A}\right)^* \left(C_1 e^{\delta A}\right) J_n \\ & = A^* - J_n^* A^* J_n + J_n^* A^* J_n + \frac{\alpha}{2} J_n^* \left(C_1 e^{\delta A}\right)^* \left(C_1 e^{\delta A}\right) J_n \end{aligned}$$

is dissipative if the parameters α, β are chosen as in (10).

This dissipativity allows to go ahead in (17) and to conclude that

$$\begin{aligned} & E\left[\left|J_n^* e^{\delta A^*} Y_t\right|^2\right] + E\left[\int_t^T \left|J_n^* e^{\delta A^*} Z_s\right|^2 ds\right] \\ & \leq \beta \left|C_2\right|^2 E\left[\int_t^T \left|Y_s\right|^2 ds\right] + \left(\frac{1}{\alpha} + \frac{1}{\beta}\right) E\left[\int_t^T \left|Z_s\right|^2 ds\right]. \end{aligned}$$

Recall that $(Y,Z) \in L^2\left(\Omega \times [0,T]; H \times L_2(\Xi; H)\right)$. Thus, letting $n \to \infty$ and then $\delta \to 0$ in the above estimate, we get

$$E\left[\left|Y_t\right|^2\right] + \left(1 - \frac{1}{\alpha} - \frac{1}{\beta}\right) E\left[\int_t^T \left|Z_s\right|^2 ds\right] \leq k\beta \left|C_2\right|^2 E\left[\int_t^T \left|Y_s\right|^2 ds\right].$$

Finally, we take the supremum over $t \in [0,T]$ and apply Gronwall's inequality. Thus we obtain

$$\sup_{t \in [0,T]} E\left[\left|Y_t\right|^2\right] + E\left[\int_0^T \left|Z_s\right|^2 ds\right] = 0,$$

and the claimed uniqueness follows as immediate consequence. ■

In order to really complete the proof of the theorem we still have to give the proof of Lemma 1.

Proof (of Lemma 1) A standard estimate for the process Φ defined in Lemma 1 gives the following for all $s, t \geq 0$:

$$
\begin{aligned}
E\left[|\Phi_t - \Phi_s|^2\right] &\leq k\left(E\left[\left|\left(e^{|t-s|A^*} - I\right)\Phi_{t\vee s}\right|^2\right]\right. \\
&\quad + E\left[\left|\int_{s\wedge t}^{s\vee t}\left(C_1 e^{(r-s)A}\right)^* Z_r\right|^2\right] \\
&\quad \left. + E\left[\left|\int_{s\wedge t}^{s\vee t} e^{(r-s)A^*} C_2^* Z_r\right|^2\right] + E\left[\int_{s\wedge t}^{s\vee t}\left|e^{(r-t)A^*} Z_r\right|^2 dr\right]\right) \\
&\leq k\left(E\left[\left|\left(e^{|t-s|A^*} - I\right)\Phi_{t\vee s}\right|^2\right]\right. \\
&\quad \left. + \left(1 + |t-s|^{1-2\gamma}\right) E\left[\int_{s\wedge t}^{s\vee t}|Z_r|^2 dr\right]\right). \qquad (18)
\end{aligned}
$$

Here k denotes a generic constant that is independent of $s, t \in [0, T]$ and can change from line to line.

Since $Z \in L^2\left(\Omega \times [0,T]; L_2\left(\Xi; H\right)\right)$, it is a direct consequence of the dominated convergence theorem that

$$
\lim_{s\to t} E\left[\int_{s\wedge t}^{s\vee t}|Z_r|^2 dr\right] = 0.
$$

It remains to show that also $E\left[\left|\left(e^{|t-s|A^*} - I\right)\Phi_{t\vee s}\right|^2\right]$ converges to zero, as $s \to t$. We first consider this limit for $t > s \uparrow t$. In this case

$$
E\left[\left|\left(e^{|t-s|A^*} - I\right)\Phi_{t\vee s}\right|^2\right] = E\left[\left|\left(e^{(t-s)A^*} - I\right)\Phi_t\right|^2\right],
$$

and the wished convergence follows from the dominated convergence theorem.

Let us now study the case in which $t < s \searrow t$. For this end we notice that, for all $s \geq t$,

$$
\begin{aligned}
& E\left[\left|\left(e^{|t-s|A^*} - I\right) \Phi_{t\vee s}\right|^2\right] \\
& \leq c\left(E\left[\left|\left(e^{(s-t)A^*} - I\right) e^{(T-s)A^*}\xi\right|^2\right]\right. \\
& + E\left[\left|\int_s^T \left(e^{(s-t)A^*} - I\right)\left(C_1 e^{(r-s)A}\right)^* Z_r dr\right|^2\right] \\
& + E\left[\left|\int_s^T \left(e^{(s-t)A^*} - I\right) e^{(r-s)A^*} C_2^* Z_r dr\right|^2\right] \\
& \left. + E\left[\left|\int_s^T \left(e^{(s-t)A^*} - I\right) e^{(r-s)A^*} Z_r dW_r\right|^2\right]\right) \\
& = I_1(s) + I_2(s) + I_3(s) + I_4(s).
\end{aligned}
\tag{19}
$$

For the first term we get from the dominated convergence theorem that

$$
I_1(s) \leq kE\left[\left|\left(e^{(s-t)A^*} - I\right)\xi\right|^2\right] \to 0 \text{ as } s \searrow t.
$$

Next,

$$
\begin{aligned}
I_2(s) \leq & \left(\int_s^T (r-s)^{-2\gamma} dr\right) \times \\
& \times E\int_t^T I_{]s,T]}(r)\left|(r-s)^{\gamma}\left(e^{(s-t)A^*} - I\right)\left(C_1 e^{(r-s)A}\right)^* Z_r\right|^2 dr.
\end{aligned}
\tag{20}
$$

We let $t < r \leq T$ and choose an arbitrary $s_0 \in \,]t, r[$. Then, for all $t < s < s_0$,

$$
\begin{aligned}
& \left|\left(e^{(s-t)A^*} - I\right)\left(C_1 e^{(r-s)A}\right)^* Z_r\right| \\
& = \left|\left(e^{(s-t)A^*} - I\right) e^{(s_0-s)A^*}\left(\left(C_1 e^{(r-s_0)A}\right)^* Z_r\right)\right| \\
& \leq k\left|\left(e^{(s-t)A^*} - I\right)\left(C_1 e^{(r-s_0)A}\right)^* Z_r\right|.
\end{aligned}
$$

Obviously, the latter expression converges to 0 as $s \searrow t$. Consequently

$$
I_{]s,T]}(r)\left|(r-s)^{\gamma}\left(e^{(s-t)A^*} - I\right) e^{(r-s)A^*} C_2^* Z_r\right|^2 \underset{s\searrow t}{\to} 0, \text{ for all } r > t,
$$

and from the dominated convergence theorem it follows that

$$
I_2(s) \to 0 \text{ as } s \searrow t.
$$

A similar argument yields $I_3(s) \to 0$ as $s \searrow t$. Finally, for the last term, we have

$$I_4(s) \leq E\left[\int_s^T \left|\left(e^{(s-t)A^*} - I\right) e^{(r-s)A^*} Z_r\right|^2 dr\right]$$
$$\leq E\left[\int_s^T \left|\left(e^{(s-t)A^*} - I\right) Z_r\right|^2 dr\right],$$

and, again by the dominated convergence theorem,

$$I_4(s) \to 0 \text{ as } s \searrow t.$$

Therefore, returning to (19) we get

$$\lim_{s\searrow t} E\left[\left|\left(e^{|t-s|A^*} - I\right) \Phi_{t\vee s}\right|^2\right] = 0.$$

This concludes the proof of our lemma.■

After having studied the existence and unique for the BSDE adjoint to our forward stochastic control problem we are able now to characterize their duality.

For the sake of simplicity, we shall assume from now on that C_1 takes its values in $L_2(\Xi; H)$.

Proposition 1 *Let $X^{x,u}$ be the unique mild solution of (4) associated to an admissible control u, and let (Y, Z) be the unique mild solution of (6). Then the following duality relation holds true*

$$E\left[\langle X_T^{x,u}, Y_T\rangle\right] = E\left[\langle x, Y_0\rangle\right] + E\left[\int_0^T \langle Bu_s, Y_s\rangle\, ds\right]. \tag{21}$$

Proof For the proof of the duality relation we have the same difficulty as in the proof of Theorem 1: we can't apply Itô's formula directly to our forward SDE and our BSDE in infinite dimensions. This is why we consider the following approximating equations

$$\begin{cases} dX_t^{n,\delta} = \left(A_n X_t^{n,\delta} + Bu_t\right) dt + \left(C_1 e^{\delta A} J_n + C_2\right) X_t^{n,\delta} dW_t, \\ X_0^n = x \in H, \end{cases}$$

and

$$\begin{cases} dY_t^{n,\delta} = -\left(A_n^* Y_t^{n,\delta} + J_n^* e^{\delta A^*} C_1^* Z_t^{n,\delta} + C_2^* Z_t^{n,\delta}\right) dt + Z_t^{n,\delta} dW_t, \\ Y_T^{n,\delta} = \xi \in L^2\left(\Omega, \mathcal{F}_T, P; H\right). \end{cases}$$

Recall that $A_n^* = n(nI - A^*)^{-1} A^* = J_n^* A^*$. To the above approximating equations we now can apply Itô's formula, and we get

$$E\left\langle Y_s^{n,\delta}, X_s^{n,\delta}\right\rangle = E\left\langle Y_t^{n,\delta}, X_t^{n,\delta}\right\rangle + E\left[\int_t^s \left\langle Bu_r, Y_r^{n,\delta}\right\rangle dr\right], \tag{22}$$

for all $0 \leq t < s \leq T$. Moreover, standard SDE and BSDE estimates show that there exists some positive constant k (not depending on δ and n), such that

$$E\left[\sup_{t\in[0,T]}\left|X_t^{n,\delta}\right|^2\right] \leq k\left(1+|x|^2\right) \text{ and}$$
$$\sup_{t\in[0,T]} E\left[\left|Y_t^{n,\delta}\right|^2\right] + E\left[\int_0^T \left|Z_s^{n,\delta}\right|^2 ds\right] \leq kE\left[|\xi|^2\right].$$

It follows that there exists some subsequence, still denoted $\left(X^{n,\delta}, Y^{n,\delta}, Z^{n,\delta}\right)$, which converges weakly to some limit $\left(X^{'}, Y^{'}, Z\right)$ in
$L^2\left(\Omega \times [0,T]; P\otimes dt; H \times H\right) \times L^2\left(\Omega\times[0,T]; P\otimes dt; L_2\left(\Xi; H\right)\right)$ as $n \to \infty$, $\delta \searrow 0$. We denote by X the continuous version of $X^{'}$; it is the unique mild solution of equation (4). Moreover, we let Y be the $dtdP$-version of Y', which belongs to $C\left([0,T]; L^2\left(\Xi; H\right)\right)$, and is, together with the process Z, the unique mild solution of (6). Moreover, from the above estimates satisfied by $\left(X^{n,\delta}, Y^{n,\delta}, Z^{n,\delta}\right)$ we get with the help of Mazur's theorem estimate (7) and

$$E\left[\sup_{t\in[0,T]}|X_t|^2\right] \leq k\left(1+|x|^2\right).$$

Moreover, if we take the weak limit as $n \to \infty$ and $\delta \searrow 0$ in (22) we get

$$E\left\langle Y_s', X_s'\right\rangle = E\left\langle Y_t', X_t'\right\rangle + E\left[\int_t^s \left\langle Bu_r, Y_r'\right\rangle dr\right],\ dtds\text{-a.e., } 0 \leq t < s \leq T.$$

Consequently,

$$E\left\langle Y_s, X_s\right\rangle = E\left\langle Y_t, X_t\right\rangle + E\left[\int_t^s \left\langle Bu_r, Y_r\right\rangle dr\right],\ \text{for all } 0 \leq t < s \leq T.$$

Finally, by taking $s = T$ and $t = 0$, we get the assertion. The proof is complete.■

The connection between equation (6) and the approximate controllability of (4) is given by the following result that generalizes those of the finite dimensional case.

Proposition 2 *(i) The linear stochastic equation (4) is approximately controllable if and only if, for every finite time horizon $T > 0$, any solution of the dual equation (6) that satisfies $B^*Y_s = 0$ dP-a.s., for all $0 \leq s \leq T$, necessarily vanishes $dsdP - a.s.$, i.e. $Y_s = 0$ dP-a.s., for all $0 \leq s \leq T$.*

(ii) The linear stochastic equation (4) is approximately null-controllable if and only if, for all finite time horizon $T > 0$, any solution of the dual equation (6) satisfying $B^*Y_s = 0$ dP-a.s., for all $0 \leq s \leq T$, is such that $Y_0 = 0$ dP-a.e.

Proof For any arbitrarily fixed time horizon $T > 0$ we get from the previous proposition that

$$E\left[\langle X_T^{x,u}, Y_T\rangle\right] = E\left[\langle x, Y_0\rangle\right] + E\left[\int_0^T \langle Bu_s, Y_s\rangle\, ds\right]. \tag{23}$$

We introduce the linear operator $M : \mathcal{U} \longrightarrow L^2(\Omega, \mathcal{F}_T, P; H)$ which associates to every admissible control u the mild solution of (4) starting from $x = 0$:

$$M(u) = X_T^{0,u} = \int_0^T e^{sA} Bu_s ds + \int_0^T e^{sA} C X_s^{0,u} dW_s.$$

Obviously, the approximate controllability (at time T) for (4) is equivalent to the condition that M has an image space dense in $L^2(\Omega, \mathcal{F}_T, P; H)$. This allows to deduce from (23) the form of the dual operator of M,

$$M^*\xi = B^*Y.$$

On the other hand, since the density of the value domain of the bounded linear operator $M \in L\left(L^2(\Omega, \mathcal{F}_T, P; H)\right)$ is equivalent with the condition that the kernel of its adjoint operator M^* is trivial, we obtain from the above relation the first assertion.

For the proof of the second assertion we introduce the operator $L : H \longrightarrow L^2(\Omega, \mathcal{F}, P; H)$ which associates to each initial state $x \in H$ the mild solution of (4) corresponding to the control $u \equiv 0$:

$$L(x) = e^{tA}x + \int_0^T e^{sA} C X_s^{x,0} dW_s.$$

From the relation $X_T^{x,u} = L(x) + M(u)$ we deduce easily that the approximate null-controllability of X is equivalent to the condition that $\overline{\mathrm{Im}}(L) \subset \overline{\mathrm{Im}}(M)$ ($\overline{\mathrm{Im}}(L), \overline{\mathrm{Im}}(M)$ are the closures of the image spaces of L and M, resp.) and hence also to the following condition:

$$Ker\,(M^*) \subset Ker(L^*).$$

On the other hand, from (23) we get $L^*\xi = Y_0$. This relation together with $M^*\xi = B^*Y = 0$ allow now to see the equivalence between the approximate null-controllability of X and the condition given in the second assertion.■

In what follows we will need the notion of the backward viability kernel introduced by Buckdahn, Quincampoix, Răşcanu [2]

Definition 1 *Let K be a nonempty, convex, closed subset of H.*

(i) A continuous stochastic process $\{Y_t,\ t \in [0,T]\}$ is called viable in K if and only if $Y_t \in K$, P-a.s., for all $t \in [0,T]$.

(ii) We say that the set K enjoys the backward stochastic viability property at time T with respect to (6) if for every K-valued terminal condition $\eta \in L^2(\Omega, \mathcal{F}_T, P; K)$, the solution $\{Y_t^\eta,\ t \in [0,T]\}$ of (6) is viable in K.

(iii) The largest closed, convex subset of K enjoying the backward stochastic viability property is called the backward stochastic viability kernel of K.

The notion of the stochastic viability kernel allows to reformulate the criterion for the approximate controllability, stated in Proposition 2:

Proposition 3 *The linear stochastic equation (4) is approximately controllable if and only if, for every finite time horizon $T > 0$, the backward stochastic viability kernel of $Ker\ B^* = \{y \in H : \ B^*y = 0\}$ at time T with respect to (6) is the trivial subspace $\{0\}$.*

Remark 3 In the finite dimensional case, the backward equation (6) may be interpreted as a forward controlled equation. Therefore, instead of studying the backward viability kernel, one may choose to investigate approximate controllability with the help of the (forward) viability kernel. Riccati methods are well adapted to control problems and allow nice characterizations of the (forward) viability kernel. The authors of [3] use these methods and show that approximate controllability of (4) is equivalent to the following invariance condition:

The largest $(A^*; C^*)$-strictly invariant linear subspace of $Ker\ B^*$ is $\{0\}$.

We recall that a linear subspace $V \subset \mathbb{R}^n$ is said to be $(A^*; C^*)$-strictly invariant if $A^*V \subset Span\{V; C^*V\} = \{\lambda v + \mu w : v \in V,\ w \in C^*V\}$.

If H is infinite dimensional, and A is a generator of a strongly continuous group, similar arguments apply.

Remark 4 Let us suppose that the Brownian motion W is 1-dimensional, $B \in \mathcal{L}(H)$, and C is a linear (possibly unbounded) operator on H such that $A^*B^* = B^*A^*$ and $B^*C^* = C^*B^*$. Then (4) is approximately controllable if and only if the image space $\mathrm{Im}(B)$ *is dense in H.*

Indeed, let us notice that if (Y, Z) is the mild solution of (6) and satisfies (7), then

$$Y_t = e^{(T-t)A^*}\xi + \int_t^T e^{(s-t)A^*} C^* Z_s ds - \int_t^T e^{(s-t)A^*} Z_s dW_s,$$

and, from the commutativity of B^* with A^* and with C^*,

$$B^*Y_t = e^{(T-t)A^*}B^*\xi + \int_t^T e^{(s-t)A^*} C^* B^* Z_s ds - \int_t^T e^{(s-t)A^*} B^* Z_s dW_s.$$

Thus, B^*Y_t is the unique mild solution of the following BSDE:

$$\begin{cases} d\widetilde{Y}_t = -A^*\widetilde{Y}_t dt - C^*\widetilde{Z}_t dt + \widetilde{Z}_t dW_t, \\ \widetilde{Y}_T = B^*\xi. \end{cases}$$

Obviously, $\widetilde{Y} = 0$ if and only if $B^*\xi = 0$ P-a.s.. Thus, from Proposition 2 it follows that Eq. (4) is approximately controllable if, for all $\xi \in L^2(\Omega, \mathcal{F}_T, P; H)$, the relation $B^*\xi = 0$, $P-a.s.$, implies that $\xi = 0$, $P-a.s.$ This is, of course, equivalent with the density of the image space $\mathrm{Im}(B)$ *in H.*

4 A necessary condition for approximate controllability

We have seen that approximate controllability for the forward controlled equation (4) is equivalent to the following (approximate) observability condition on the dual equation (6) :

$$\text{"}B^*Y_t = 0,\ dP - a.s.,\ \text{for all } t \in [0,T],\ \text{implies } Y_T = 0,\ dP - a.s.\text{"} \quad (24)$$

In the deterministic case, Russell and Weiss [20] generalized the Hautus test of observability for infinite dimensional equations with an operator A that is supposed to generate an exponentially stable semigroup. In what follows we assume besides (A1) and (A2) the following additional condition:

(A3) *The linear operator A generates an exponentially stable, strongly continuous semigroup of operators.*

Under the assumptions (A1)-(A3) we can prove the following statement:

Proposition 4 *A necessary condition for the approximate controllability of (4) is that, for every $y \in D(A^*)$ and every $\alpha < 0$,*

$$|B^*y| + |(A^* - \alpha I)\, y| > 0,\ \textit{whenever } y \neq 0. \quad \text{(N1)}$$

Proof In order to prove the claim, let us first notice that $H_1 = D(A)$ endowed with the norm $|h|_1 = |(A^* - \alpha I)\, h|_H$ is a Hilbert space. It is well known that, under the above assumptions, the family of norms indexed by $\alpha < 0$ are equivalent with the usual graph norm on H_1. For every $y \in D(A^*)$ we let (Y^y, Z^y) denote the unique mild solution in H of the BSDE

$$\begin{cases} dY_t^y = -A^*Y_t^y dt - C^*Z_t^y dt + Z_t^y dW_t, \\ Y_T = y. \end{cases}$$

Since all data of this BSDE is deterministic it is immediate that Y^y is deterministic and $Z^y = 0$. In particular, we see that $Y_t^y = e^{(T-t)A^*}y$ is a classical solution (in H) of

$$\begin{cases} dY_t^y = -\alpha Y_t^y dt - e^{(T-t)A^*}\,(A^* - \alpha I)\, y dt, \\ Y_T^\eta = y, \end{cases}$$

and the function B^*Y^y is a classical solution of the following equation:

$$\begin{cases} d\,(B^*Y_t^y) = -\alpha\,(B^*Y_t^y)\, dt - B^*e^{(T-t)A^*}\,(A^* - \alpha I)\, y dt \\ B^*Y_T^y = B^*y. \end{cases}$$

It follows easily from this equation that $B^*Y_t^y = 0$, for all $t \in [0,T]$, if and only if

$$\begin{cases} B^*y = 0, \\ B^*e^{tA^*}\,(A^* - \alpha I)\, y = 0,\ \text{for all } t \in [0,T]. \end{cases}$$

Consequently, the condition (24) gives the following necessary condition for the approximate controllability of (4):

$$\text{"}B^*Y_t^y = 0,\ \text{for all } t \in [0,T],\ \text{implies } y = 0.\text{ "}$$

Obviously, the two latter conditions allow to conclude that

$$\begin{cases} B^* y = 0, \\ B^* e^{tA^*} (A^* - \alpha I) y = 0, \text{ for all } t \in [0,T], \end{cases} \text{implies } y = 0, \qquad (25)$$

and the estimate

$$\left| B^* e^{tA^*} (A^* - \alpha I) y \right| \le k \left| (A^* - \alpha I) y \right|,$$

in combination with (25) allows to complete the proof.■

Remark 5 Jacob, Partington [13] studied the approximate controllability for a deterministic system. They supposed

(JP) A is an infinitesimal generator of an exponentially stable, strongly continuous semigroup which possesses a sequence of normalized eigenvectors $\{e_i\}$ corresponding to the eigenvalues $\{\lambda_i\}$ such that $\sup_i \lambda_i < 0$. Moreover, they considered the case of a 1-dimensional input space, i.e. $B \in L(\mathbb{R}; H)$.

In this particular case, the necessary and sufficient condition for approximate controllability of the deterministic system

$$\begin{cases} dX_t^{x,u} = (AX_t^{x,u} + Bu_t)\, dt, \\ X_0 = x \in H, \end{cases}$$

found by the authors, says that for all $y \in H_1$ and all $\alpha < 0$,

$$|B^* y|^2 + |(A^* - \alpha I) y|^2 > 0 \text{ whenever } y \neq 0.$$

Remark 6 For the case in which H is n-dimensional Euclidean space (stochastic) approximate controllability was studied by Buckdahn, Quincampoix, Tessitore [3] and Goreac [11]. The equivalent condition for approximate controllability reads

$$\text{The largest } (A^*; C^*)\text{-strictly invariant subspace of } Ker\ B^* \text{ is } \{0\}. \qquad (26)$$

Let us suppose that, for the framework studied by these authors, there exists a bounded linear operator $D \in L(U)$ such that $B^* C^* = DB^*$. Then we get that $Ker\ B^*$ is C^*- invariant, and thus (26) can be written as follows:

$$\text{The largest } A^*\text{-invariant subspace of } Ker\ B^* \text{ is } \{0\}. \qquad (27)$$

Moreover, under the assumptions of Jacob, Partington [13] (JP), it is obvious that (N1) is equivalent to (27). Indeed, if (N1) holds true, then

$$\begin{cases} B^* e_i \neq 0, \text{ for all } 1 \le i \le n, \\ \lambda_i \neq \lambda_j, \text{ for all } 1 \le i, j \le n, \ i \neq j. \end{cases}$$

(see Jacob, Partington [13], Theorem 4.1). Let V denote the largest A^*-invariant subspace of $Ker\ B^*$, and let us suppose that there exists some linear combination $v = \sum_{k=1}^m v_{i_k} e_{i_k}$ such that $v \in V$, where $m \le$

$n, i_k \in \{1, 2, \ldots, n\}$ and $v_{i_k} \neq 0$, for all $1 \leq k \leq m$. Then, for all $j \geq m-1$, $\sum_{k=1}^{m} \lambda_{i_k}^{j} v_{i_k} e_{i_k} \in V$. Thus, since

$$\det \left[\lambda_{i_k}^{j} v_{i_k}\right]_{k,j} = \prod_{1\leq k\leq m} v_{i_k} \prod_{1\leq k<j\leq m} \left(\lambda_{i_j} - \lambda_{i_k}\right) \neq 0,$$

we get that

$$span\left\{e_{i_k}, 1 \leq k \leq m\right\} \subset V.$$

It follows that $V = span\left\{e_{i_k}, 1 \leq k \leq N\right\}$, for some $N \leq n$. But then $B^* e_{i_k} = 0$, and this contradicts our assumption and we have that $V = \{0\}$.

For the converse, if (27) holds true and $y \in H_1$ such that

$$|B^* y|^2 + |(A^* - \alpha I)\, y|^2 = 0 \text{ for some } \alpha < 0,$$

then $V = span\{y\}$ is A^*-invariant and included in $Ker\ B^*$. It follows that $y = 0$, and we get (N1). This latter argument applies also when H has infinite dimension.

Let us now make the following assumptions:

(B) *W is supposed to be a 1-dimensional Brownian motion, the control state space U is a bounded closed subspace of some separable real Hilbert space V, $B \in \mathcal{L}(V; H)$, A is a self adjoint operator which generates a semigroup of contractions on H, and the operator C admits a decomposition*

$$C = C_1 + C_2,$$

of two linear operators C_1, C_2 which are supposed to have the following properties:

1) C_2 is a bounded operator from H to H;

2) for all $t > 0$, $C_1 e^{tA}$, $e^{tA} C_1 \in \mathcal{L}(H)$. Moreover, we suppose that there exist some $\gamma \in \left[0, \frac{1}{2}\right)$ and some positive constant $L > 0$ such that

$$\left|C_1 e^{tA}\right|_{\mathcal{L}(H)} + \left|e^{tA} C_1\right|_{\mathcal{L}(H)} \leq L t^{-\gamma},$$

for all $t > 0$.

3) There exists some constant $a > \frac{1}{2}$ such that

$$A + a C_1^* C_1 \text{ is dissipative.}$$

We recall the following

Definition 2 *Let A be the generator of a C_0-semigroup on the Hilbert space H and C is a linear operator on H. We say that C is a class-$\mathcal{P}$ perturbation of A if C is closed,*

$$D(C) \supset \cup_{t>0} e^{tA}(H) \;\; and \;\; \int_0^1 \left|Ce^{tA}\right| dt < \infty.$$

Obviously, under the above assumptions, the operator C is a class-$\mathcal{P}$ perturbation of A. It follows that $A+\lambda C$ is the generator of a C_0-semigroup $\left(e^{t(A+\lambda C)}\right)_{t\geq 0}$ for all $\lambda \in \mathbb{R}$ (cf. Davies [7] Theorem 3.5).

For the study of the main result of this section we will need the following estimates:

Lemma 2 *Under our standard assumptions we have that, for some constant k,*

$$\left|C_1 e^{t(A+\lambda C)}\right|_{\mathcal{L}(H)} + \left|e^{t(A+\lambda C)} C_1\right|_{\mathcal{L}(H)} \leq k\left(t^{-\gamma}+1\right),$$

for all $t \in [0,T]$.

Proof From the theory of general perturbation of generators it follows that

$$\begin{aligned} e^{t(A+\lambda C)}x = e^{tA}x + \lambda \int_0^t e^{(t-s)A} C_1 e^{s(A+\lambda C)} x \\ + \lambda \int_0^t e^{(t-s)A} C_2 e^{s(A+\lambda C)} x, \end{aligned}$$

for all $x \in H$. Then, by applying on both sides of the above relation the bounded operator C_2, we get the following norm estimate:

$$\begin{aligned} \left|C_1 e^{t(A+\lambda C)} x\right| \leq t^{-\gamma}|x| + \lambda \int_0^t (t-s)^{-\gamma}\left|C_1 e^{s(A+\lambda C)} x\right| ds \\ + k\int_0^t (t-s)^{-\gamma}|x|\, ds, \end{aligned}$$

for all $t \in [0,T]$. Here k denotes again a generic constant which can depend on λ and T. Thus, applying Cauchy-Schwarz inequality yields

$$\begin{aligned} \left|C_1 e^{t(A+\lambda C)} x\right|^2 &\leq k\left(\left(t^{-2\gamma}+t^{2-2\gamma}\right)|x|^2 + t^{1-2\gamma}\int_0^t \left|C_1 e^{s(A+\lambda C)} x\right|^2 ds\right) \\ &\leq k\left(\left(t^{-2\gamma}+1\right)|x|^2 + \int_0^t \left|C_1 e^{s(A+\lambda C)} x\right|^2 ds\right), \end{aligned}$$

and from Gronwall's inequality we finally get

$$\left|C_1 e^{t(A+\lambda C)} x\right|^2 \leq k\left(t^{-\gamma}+1\right)^2 |x|^2 .$$

It follows that $C_1 e^{t(A+\lambda C)} \in \mathcal{L}(H)$ and

$$\left|C_1 e^{t(A+\lambda C)}\right|_{\mathcal{L}(H)} \leq k\left(t^{-\gamma}+1\right),$$

for all $t \in [0,T]$. Using a similar argument we can prove that $e^{t(A+\lambda C)} C_1 \in \mathcal{L}(H)$ and

$$\left|e^{t(A+\lambda C)} C_1\right|_{\mathcal{L}(H)} \leq k\left(t^{-\gamma}+1\right),$$

for all $t \in [0,T]$. ■

To establish the main result of this section we shall further introduce the following set standing for the joint dissipativity condition on A, C:

$$\Lambda = \left\{ \lambda \in \mathbb{R} : \exists a > \frac{1}{2} \text{ such that } A + \lambda C_1 + a C_1^* C_1 \text{ is dissipative} \right\}.$$

Remark 7 1. If $C \in \mathcal{L}(H)$ is a bounded operator, then $\Lambda = \mathbb{R}$.
2. Λ contains at least the origin $\{0\}$.
3. If C_1 is dissipative and the assumption (B) holds true, then $\mathbb{R}_+ \subset \Lambda$.

We now can state our main result of this section.

Theorem 2 *Under assumption (B), a necessary condition for the approximate controllability of (4) is*

$$|B^* y| + |(A^* + \lambda C^* - \alpha I) y| > 0, \text{ for all } y \neq 0, \text{ and all } (\lambda, \alpha) \in \Lambda \times \mathbb{R}_-. \tag{28}$$

The above necessary condition is an immediate consequence of Proposition 4 and a λ-wise application of the following result:

Theorem 3 *If (4) is approximately controllable, then the system*

$$\begin{cases} dX_t = ((A + \lambda C) X_t + B v_t) dt + (C + \lambda I) X_t dW_t, \\ X_0 = x \in H, \end{cases} \tag{29}$$

which is governed by the control process $v \in L^2_{\mathcal{F}}([0, T]; V)$ is also approximately controllable.

Proof Step 1. Approximation of (29) by an equation with bounded operators admitting the application of Itô's formula.

For all $u \in L^0_{\mathcal{F}}([0, T]; U)$, we denote by $X^{x,u}_{n,\delta}$ the unique mild solution of the controlled forward equation

$$\begin{cases} dX^{x,u}_{n,\delta}(t) = A_n X^{x,u}_{n,\delta}(t) dt + Bu(t) dt + J_n^* e^{\delta A^*} C e^{\delta A} J_n X^{x,u}_{n,\delta}(t) dW_t, \\ X^{x,u}_{n,\delta}(0) = x \in H, \end{cases}$$

where $J_n = \left(I - n^{-1} A\right)^{-1}$ and $A_n = J_n A$. This approximation of the operators A (by A_n) and C (by $J_n^* e^{\delta A^*} C e^{\delta A} J_n$) explains by the same difficulties we have already met in the proof of Theorem 1. Our special choice of the approximation allows to conserve the joint dissipativity condition also for the approximating operators and allows now to apply Itô's formula.

Let $\mathcal{E}(\lambda W)$ denote the Doléan-Dade exponential of λW, i.e., $\mathcal{E}(\lambda W)_t := e^{\lambda W_t - \frac{\lambda^2}{2} t}$, $t \in [0, T]$. Then, from Itô's formula applied to $\mathcal{E}(\lambda W)_t X^{x,u}_{n,\delta}(t)$ it follows that

$$\begin{cases} d\left(\mathcal{E}(\lambda W)_t X^{x,u}_{n,\delta}(t)\right) = \left(A_n + \lambda J_n^* e^{\delta A^*} C e^{\delta A} J_n\right) \left(\mathcal{E}(\lambda W)_t X^{x,u}_{n,\delta}(t)\right) dt \\ \qquad\qquad + B\left(\mathcal{E}(\lambda W)_t u(t)\right) dt \\ \qquad\qquad + \left(J_n^* e^{\delta A^*} C e^{\delta A} J_n + \lambda I\right) \left(\mathcal{E}(\lambda W)_t X^{x,u}_{n,\delta}(t)\right) dW_t, \\ X^{x,u}_{n,\delta}(0) = x \in H. \end{cases}$$

After the above application of Itô's formula we would like to take the limit as $n \to +\infty$ and then as $\delta \downarrow 0$ in order to get an equation which coincides with that we would get if we applied formally Itô's formula to $\mathcal{E}(\lambda W)_t X^{x,u}(t)$, where $X^{x,u}$ denotes the unique mild solution of (4). For taking these limits we need the following result whose proof will be given later.

Proposition 5 *Under the assumptions on Theorem 2 and with the notations introduced above we have that, for all $x \in H$,*

$$\lim_n \sup_{0 \le t \le T} \left| e^{t\left(A_n + \lambda J_n^* e^{\delta A^*} C e^{\delta A} J_n\right)} x - e^{t\left(A + \lambda e^{\delta A^*} C e^{\delta A}\right)} x \right| = 0, \; \delta > 0, \tag{30}$$

and

$$\lim_\delta \sup_{0 \le t \le T} \left| e^{t\left(A + \lambda e^{\delta A^*} C e^{\delta A}\right)} x - e^{t(A+\lambda C)} x \right| = 0. \tag{31}$$

We continue the

Proof of our theorem. With the help of the above proposition we are now able to prove

Step 2. Let $X^{x,u}$ denote the unique mild solution of (4). Then the process $\mathcal{E}(\lambda W)_\cdot X^{x,u}(\cdot)$ is the unique mild solution of (29). Moreover,

$$\sup_{0 \le t \le T} E\left[\left| \mathcal{E}(\lambda W)_t X^{x,u}(t) \right|^p \right] \le c_p \left(1 + |x|^p\right). \tag{32}$$

For proving this statement we first notice that from standard estimates, for all $p > 2$,

$$E\left[\sup_{0 \le t \le T} \left| \mathcal{E}(\lambda W)_t X_{n,\delta}^{x,u}(t) \right|^p \right] \le c_p \left(1 + |x|^p\right), \text{ and}$$
$$E\left[\sup_{0 \le t \le T} \left| X_{n,\delta}^{x,u}(t) \right|^p \right] \le c_p \left(1 + |x|^p\right);$$

c_p denotes a generic constant independent of n, δ and $u \in L^0_{\mathcal{F}}([0,T];U)$. Then, for any $\delta > 0$, there exists a subsequence of $\left(\mathcal{E}(\lambda W)_\cdot X_{n,\delta}^{x,u}(\cdot), X_{n,\delta}^{x,u}(\cdot)\right)_n$, still denoted by $\left(\mathcal{E}(\lambda W)_\cdot X_{n,\delta}^{x,u}(\cdot), X_{n,\delta}^{x,u}(\cdot)\right)_n$,which converges in the weak topology on $L^p([0,T] \times \Omega; H) \times L^{2p}([0,T] \times \Omega; H)$ to some limit $\left(X'_\delta(\cdot), X''_\delta(\cdot)\right)$. With the help of Proposition 5 we can show that X'_δ is a unique mild solution of

$$\begin{cases} dX'_\delta(t) = \left(A + \lambda e^{\delta A^*} C e^{\delta A}\right) X'_\delta(t) dt \\ \qquad\qquad + B\left(\mathcal{E}(\lambda W)_t u(t)\right) dt + \left(e^{\delta A^*} C e^{\delta A} + \lambda I\right) X'_\delta(t) dW_t, \\ X'_\delta(0) = x \in H, \end{cases}$$

and X''_δ is a mild solution of

$$\begin{cases} dX''_\delta(t) = (AX''_\delta(t) + Bu_t)\,dt + e^{\delta A^*} C e^{\delta A} X''_\delta(t) dW_t, \\ X''_\delta(0) = x \in H. \end{cases} \tag{33}$$

On the other hand, it follows from the general theory of SDEs in infinite dimensions that these mild solutions are unique and that

$$\sup_{0\le t\le T} E\left[|X'_\delta(t)|^p\right] \le c_p\left(1+|x|^p\right). \tag{34}$$

Moreover, taking into account that
$\mathcal{E}(\lambda W)_\cdot\, \zeta(\cdot) \in L^{\frac{2p}{2p-1}}([0,T]\times\Omega;H)$, for all $\zeta \in L^{\frac{p}{p-1}}([0,T]\times\Omega;H)$, we get

$$E\left[\int_0^T \langle X'_\delta(t), \zeta(t)\rangle\,dt\right] = \lim_n E\left[\int_0^T \left\langle \mathcal{E}(\lambda W)_t\, X^{x,u}_{n,\delta}(t), \zeta(t)\right\rangle dt\right]$$

$$= \lim_n E\left[\int_0^T \left\langle X^{x;u}_{n,\delta}(t), \mathcal{E}(\lambda W)_t\,\zeta(t)\right\rangle dt\right]$$

$$= E\left[\int_0^T \langle \mathcal{E}(\lambda W)_t\, X''_\delta(t), \zeta(t)\rangle\,dt\right].$$

This relation allows to identify the processes $X'_\delta(\cdot)$ and $\mathcal{E}(\lambda W_\cdot)\, X''_\delta(\cdot)$ as elements of $L^p([0,T]\times\Omega;H)$. Moreover, if $X^{x,u}_\delta$ denotes the continuous version of X''_δ and $\widetilde{X}^{x,u}_\delta$the continuous version of X'_δ, we have

$$\widetilde{X}^{x,u}_\delta(t) = \mathcal{E}(\lambda W)_t\, X^{x,u}_\delta(t),\ dP\text{-a.s, for all } t\in[0,T],$$

and inequality (34) takes the form

$$\sup_{0\le t\le T} E\left[|\mathcal{E}(\lambda W)_t\, X^{x,u}_\delta(t)|^p\right] \le c_p\left(1+|x|^p\right).$$

By repeating the argument for letting $\delta\to 0$ we get the result stated in step 2.

After having related equation (4) with equation (29) we can prove now the theorem in its proper sense.

Step 3. Conclusion.

If $\xi \in L^2(\Omega,\mathcal{F}_T,P;H)$, then, for every $\varepsilon>0$ there exists some $\xi^\varepsilon \in L^\infty(\Omega,\mathcal{F}_T,P;H)$ such that

$$E\left[|\xi^\varepsilon - \xi|^2\right] \le \varepsilon.$$

It follows from (32) that the family
$\left\{|\mathcal{E}(\lambda W)_T\, X^{x,u}(T) - \xi^\varepsilon|^2,\ u\in L^0_{\mathcal{F}}([0,T];U)\right\}$

is uniformly integrable. Consequently, there exists $M_\varepsilon > 0$ such that

$$E\left[\left|\mathcal{E}(\lambda W)_T X^{x,u}(T) - \xi^\varepsilon\right|^2 1_{\left\{\mathcal{E}(\lambda W)_T > M_\varepsilon\right\}}\right] \le \varepsilon,$$

for all $u \in L^0_{\mathcal{F}}([0,T];U)$. If the equation (4) is approximately controllable, then there exists $u_\varepsilon \in L^0_{\mathcal{F}}([0,T];U)$ such that

$$E\left[\left|X_T^{x,u_\varepsilon} - \xi^\varepsilon \mathcal{E}(\lambda W)_T^{-1}\right|^2\right] \le \frac{\varepsilon}{M_\varepsilon^2},$$

and we get

$$\begin{aligned} E\left[\left|\mathcal{E}(\lambda W)_T X_T^{x,u_\varepsilon} - \xi^\varepsilon\right|^2\right] &\le M_\varepsilon^2 E\left[\left|X_T^{x,u_\varepsilon} - \xi^\varepsilon \mathcal{E}(\lambda W)_T^{-1}\right|^2\right] \\ &\quad + E\left[\left|\mathcal{E}(\lambda W)_T X^{x,u_\varepsilon}(T) - \xi^\varepsilon\right|^2 1_{\left\{\mathcal{E}(\lambda W)_T > M_\varepsilon\right\}}\right] \\ &\le 2\varepsilon. \end{aligned}$$

Therefore, also (29) is approximately controllable. The proof of our theorem is now complete.■

However, the proof of Proposition 5 still remains open:

Proof (of Proposition 5). Due to the definition of the approximation of the operators A and C given in step 1 of the proof of the above theorem we have for all $x \in \mathcal{D}\left(A + \lambda e^{\delta A^*} C e^{\delta A}\right)$,

$$\lim_n \left(A_n + \lambda J_n^* e^{\delta A^*} C e^{\delta A} J_n\right) x = \left(A + \lambda e^{\delta A^*} C e^{\delta A}\right) x. \tag{35}$$

For all n, the operator $A_n + \lambda J_n^* e^{\delta A^*} C e^{\delta A} J_n$ is bounded. Therefore, it generates a C_0-semigroup $\left(e^{t\left(A_n + \lambda J_n^* e^{\delta A^*} C e^{\delta A} J_n\right)}\right)_t$ and the application $t \longmapsto \left|e^{t\left(A_n + \lambda J_n^* e^{\delta A^*} C e^{\delta A} J_n\right)}\right|$ is continuous. From the general theory of perturbation of generators, we have

$$\begin{aligned} e^{t\left(A_n + \lambda J_n^* C e^{\delta A} J_n\right)} x &= e^{tA_n} x \\ &\quad + \lambda \int_0^t e^{(t-s)A_n} J_n^* e^{\delta A^*} C_1 e^{\delta A} J_n e^{s\left(A_n + \lambda J_n^* e^{\delta A^*} C e^{\delta A} J_n\right)} x ds \\ &\quad + \lambda \int_0^t e^{(t-s)A_n} J_n^* e^{\delta A^*} C_2 e^{\delta A} J_n e^{s\left(A_n + \lambda J_n^* e^{\delta A^*} C e^{\delta A} J_n\right)} x ds. \end{aligned}$$

It follows that, for n great enough

$$\left|e^{t\left(A_n + \lambda J_n^* C e^{\delta A} J_n\right)}\right| \le 1 + \lambda \int_0^t \left(\delta^{-\gamma} + k\right) \left|e^{s\left(A_n + \lambda J_n^* e^{\delta A^*} C e^{\delta A} J_n\right)}\right| ds,$$

where $k > 0$ is a generic constant (which may depend on δ but not on n), and Gronwall's inequality yields

$$\left| e^{t\left(A_n + \lambda J_n^* e^{\delta A^*} C e^{\delta A} J_n\right)} \right| \leq e^{kt}, \tag{36}$$

for all $t > 0$, and all $n \in \mathbb{N}$. Then, from (35) and (36) we get (cf. Davies [7] Th. 3.17) that (30) holds true, for all $\delta > 0$ and all $x \in \mathcal{D}(A)$.

To prove the second assertion, we notice that

$$\begin{aligned} e^{t\left(A+\lambda e^{\delta A^*} C e^{\delta A}\right)} x = e^{tA} x &+ \int_0^t e^{(t-s)A} \lambda e^{\delta A^*} C_1 e^{\delta A} e^{s\left(A+\lambda e^{\delta A^*} C e^{\delta A}\right)} x ds \\ &+ \int_0^t e^{(t-s)A} \lambda e^{\delta A^*} C_2 e^{\delta A} e^{s\left(A+\lambda e^{\delta A^*} C e^{\delta A}\right)} x ds, \end{aligned}$$

for all $x \in H$. Then, recalling that A is self adjoint,we obtain

$$\begin{aligned} \left| e^{t\left(A+\lambda e^{\delta A^*} C e^{\delta A}\right)} x \right| \leq |x| &+ \lambda \int_0^t \left| e^{s\left(A+\lambda e^{\delta A^*} C e^{\delta A}\right)} x \right| (t-s)^{-\gamma} \, ds \\ &+ k \int_0^t \left| e^{s\left(A+\lambda e^{\delta A^*} C e^{\delta A}\right)} x \right| ds \end{aligned}$$

(k is again a generic constant independent of δ). Thus, with the notation

$$f(t) = \left| e^{t\left(A+\lambda e^{\delta A^*} C e^{\delta A}\right)} x \right|,$$

the latter estimate takes the form

$$f(t) \leq |x| + \lambda \int_0^t f(s) \, (t-s)^{-\gamma} \, ds + k \int_0^t f(s) ds.$$

Then, by Cauchy-Schwarz inequality,

$$f(t) \leq |x| + k \left(\frac{t^{\frac{1-2\gamma}{2}}}{\sqrt{1-2\gamma}} + t^{\frac{1}{2}} \right) \left(\int_0^t f^2(s) ds \right)^{\frac{1}{2}} \tag{37}$$

$$\leq |x| + k \left(T^{\frac{1}{2}} \vee 1 \right) \left(\int_0^t f^2(s) ds \right)^{\frac{1}{2}}, \tag{38}$$

and, consequently,

$$f^2(t) \leq 2 \left(|x|^2 + k \, (T \vee 1) \int_0^t f^2(s) ds \right).$$

To the latter estimate we apply Gronwall's inequality and take the square root after. This yields

$$f(t) \leq \sqrt{2} \, |x| \, e^{k(T \vee 1)t}.$$

Therefore, from the definition of $f(t)$ it follows that

$$\sup_{\delta>0} \left| e^{t\left(A+\lambda e^{\delta A^*} C e^{\delta A}\right)} \right| \leq M e^{ct}, \tag{39}$$

for all $t \leq T$, where M and c are positive constants that are independent of $\delta > 0$. On the other hand, for all $x \in \mathcal{D}(A + \lambda C)$ we have

$$\lim_{\delta \to 0} \left(A + \lambda e^{\delta A^*} C e^{\delta A} \right) x = (A + \lambda C)\, x. \tag{40}$$

The second assertion follows (cf. Davies [7] Th. 3.17).■

In the following we discuss tow examples to illustrate the results of this section.

Example 1 Given a regular domain $\mathcal{O} \subset \mathbb{R}^N$ we consider the following stochastic partial differential equation

$$\begin{cases} d_t X^u(t,x) = \sum_{i,j=1}^N \partial_i \left(a_{i,j}(x) \partial_j X^u(t,x) \right) dt + u(t)\, b(x) dt \\ \qquad\qquad + \sum_{i=1}^N c_i(x) \partial_i X^u(t,x) dW_t, \\ X^u(t,x) = 0,\ \forall\, (t,x) \in [0,T] \times \partial\mathcal{O}, \\ X^u(0,x) = \xi(x),\ \forall x \in \mathcal{O}, \end{cases} \tag{41}$$

where u is an admissible control process taking its values in $\mathbb{R}$. We suppose that $a(x)\,(= (a_{i,j}(x)))\,\sigma(x)\sigma^*(x)$ for some $C_{\ell,b}^{\infty}$ matrix σ of $N \times N$-type, $c = (c_1, \ldots, c_N) \in C_{\ell,b}^{\infty}\left(\mathcal{O}; \mathbb{R}^N\right)$, $b \in H^1(\mathcal{O})$ and $\xi \in L^2\left(\Omega, \mathcal{F}_T, P; L^2(\mathcal{O})\right)$. Moreover, we suppose that the couple of coefficients (a, c) satisfies the standard ellipticity condition

$$\sum_{i,j=1}^N \left(a_{i,j}(x) - \alpha c_i(x) c_j(x) \right) \lambda_i \lambda_j \geq 0, \tag{42}$$

for some $\alpha > \frac{1}{2}$ and for all $\lambda \in \mathbb{R}^N$. Then, if we put

$$\begin{aligned} &H = L^2(\mathcal{O}), \\ &\mathcal{D}(A) = H^2(\mathcal{O}) \cap H_0^1(\mathcal{O}),\ A\zeta = \sum_{i,j=1}^N \partial_i \left(a_{i,j}(x) \partial_j \zeta(x) \right), \\ &\mathcal{D}(C) = H^1(\mathcal{O}),\ C\zeta = c \cdot \nabla\zeta, \end{aligned}$$

we get that

$$\mathcal{D}(C^*) = H^1(\mathcal{O}),\ C^*\zeta = -c \cdot \nabla\zeta - \zeta \sum_{i=1}^N \partial_i c_i.$$

The ellipticity condition (42) insures that the dual backward stochastic partial differential equation

$$\begin{cases} d_tY(t,x) = -\left(\sum_{i,j=1}^{N} \partial_i \left(a_{i,j}(x)\partial_j Y\left(t,x\right)\right)\right) dt + \left(\sum_{i=1}^{N} c_i(x)\partial_i Z(t,x)\right) dt \\ \qquad\qquad + \left(\sum_{i=1}^{N} \partial_i c_i(x) Z(t,x)\right) dt + Z\left(t,x\right) dW_t, \\ Y(t,x) = Z(t,x) = 0, \ \forall \left(t,x\right) \in [0,T] \times \partial\mathcal{O}, \\ Y(T,x) = \eta(x), \ \forall x \in \mathcal{O}, \end{cases} \tag{43}$$

has a unique mild solution. Thus we know that the approximate controllability of (41) is equivalent to the approximate observability of (43).

From (N1) it follows that, if (41) is approximately controllable and if $\zeta_n(x)$ is a complete orthonormal base consisting of eigenvectors for A, then every coefficient of b in this base must be non null.

Remark 8 The problem of controllability for the deterministic version of (41) has been treated by Carleman estimates method in Fursikov, Imanuvilov [10].

The condition (N2) is non trivially more general then (N1) as proven by the following

Example 2 We consider the following equation

$$\begin{cases} d_tX^u\left(t,x\right) = \triangle X^u\left(t,x\right) dt + u\left(t\right) b\left(x\right) dt \\ \quad + \left(2\sin\left(\pi x\right) \int_0^1 X^u\left(t,y\right) \sin\left(\pi y\right) dy\right) dW_t, \\ X^u\left(t,0\right) = X^u\left(t,1\right) = 0, \ \forall t \in [0,T]. \\ X^u\left(0,x\right) = \xi\left(x\right), \ \forall x \in \left(0,1\right), \end{cases} \tag{44}$$

where u is an admissible real-valued bounded control process and $b \in L^2\left(0,1\right)$. This equation can be expressed as an infinite dimensional linear equation. For this we put

$$\begin{aligned} &H = L^2\left(0,1\right), \ \mathcal{D}(A) = H^2\left(0,1\right) \cap H_0^1\left(0,1\right), \\ &A\zeta = \triangle\zeta, \ \text{for all } \zeta \in \mathcal{D}(A), \\ &C\zeta\left(\cdot\right) = 2\sin\left(\pi\cdot\right) \int_0^1 \zeta\left(y\right) \sin\left(\pi y\right) dy, \ \text{for all } \zeta \in H. \end{aligned}$$

Obviously C is a self-adjoint bounded linear operator on H. Furthermore, suppose that

$$b_n = \sqrt{2} \int_0^1 b\left(y\right) \sin\left(\pi y\right) dy \neq 0,$$

for all $n \geq 1$. Then (N1) is obviously satisfied. However, if we choose $\lambda = -3\pi^2$, $\alpha = -4\pi^2$ and $\zeta\left(\cdot\right) = -\frac{b_2\sqrt{2}}{b_1} \sin\left(\pi\cdot\right) + \sqrt{2}\sin\left(2\pi\cdot\right)$, we have

$$\left|\left(A^* + \lambda C^* - \alpha I\right)\zeta\right|^2 + \left|B^*\zeta\right|^2 = 0.$$

It follows that (N2) is not satisfied which implies that the equation (44) cannot be approximately controllable.

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Insurance, Reinsurance and Dividend Payment

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Abstract

The aim of this paper is to introduce an insurance model allowing reinsurance and dividend payment. Our model deals with several homogeneous contracts and takes into account the legislation regarding the provisions to be justified by the insurance companies. This translates into some restriction on the (maximal) number of contracts the company is allowed to cover. We deal with a controlled jump process in which one has free choice of retention level and dividend amount. The utility function is given as the maximized expected discounted dividends. We prove that this value function is a viscosity solution of some first-order Hamilton-Jacobi-Bellman variational inequality. Moreover, a uniqueness result is provided.

Key words: Stochastic control, jump diffusion, viscosity solution, insurance, reinsurance

1 Preliminaries

A common problem of the insurance companies is to find a strategy allowing to satisfy the claims appearing either from the insured parties as consequence to specified peril or from the shareholders in terms of dividends. To reduce their risks and protect themselves from very large losses, the companies usually choose to pay some of the premiums to a third party. This process is called reinsurance, and it commits the third party (the reinsurance company) to cover a certain part of the claims. It is obvious that the insurance company controls the contracts to be reinsured as well as the dividends to be paid to the shareholders. These elements justify the framework of stochastic control.

This paper considers a utility function given as the maximized expected discounted dividends. In the literature, this approach has been first used by Jeanblanc, and Shiryaev (1995). In their model, the capital of an insurance company is described with the help of a standard Brownian motion and the dividend payment strategy is understood as control process. More precisely, they deal with the following model

$$dX_t = \mu dt + \sigma dW_t - dZ_t,$$

where μ and σ are arbitrary constants, W is a 1-dimensional standard Brownian motion and Z is an adapted, non decreasing, right-continuous process which represents the dividend payment strategy.

In Asmussen et al. (2000), a model concerning excess-of-loss reinsurance and dividend payment has been studied. They use diffusion and proportional reinsurance for their model. More exactly, they take as model of the capital of the insurance company the process given by the following equation

$$dX_t = a_t\left(\mu dt + \sigma dW_t\right) - dZ_t,$$

where $0 \leq a_t \leq 1$ stands for the retention level. In the case where the rate of dividend pay-out is unrestricted, they characterize the value function as the (classical) solution of some associated Hamilton-Jacobi-Bellman equation.

The same problem is studied by Mnif, Sulem (2005), but the claims are represented by a compound Poisson process. In their collective risk model, a retention level is an adapted process α_t which specifies that, for a claim y, the direct insurer covers $y \wedge \alpha_t$, while the reinsurance company covers the remaining $(y - \alpha_t)^+$. They consider a single insurance contract and the reserve of the insurance company satisfies

$$dX_t = p(\alpha_t)dt - \int_B (y \wedge \alpha_t)\,\mu(dtdy) - dL_t,$$

where μ is the random measure associated to the compound Poisson process, $p(\alpha_t)$ is the actual premium of the insurance company given the retention level α, after reinsurance of the excess of loss, and L_t is an adapted, càdlàg process such that $L_t - L_{t-} \leq X_{t-}$ for all $t \geq 0$. The process L describes the pay-out of dividends for shareholders. The value function is defined as the maximized expected discounted dividends until the ruin time τ,

$$V(x) = \sup_{(u,L)} E\left[\int_0^\tau e^{-rs} dL_s\right];$$

here r is some positive discount factor. The authors proved that, under the assumption that the value function satisfies the dynamic programming principle, V is a viscosity solution of the associated Hamilton-Jacobi-Bellman variational inequality.

In the present paper we consider the problem of optimal reinsurance and dividend pay-out with several insurance contracts. We will prove that in the framework of the collective risk model, even if the invested initial capital is arbitrarily small, one can expect a gain which exceeds an a priori fixed positive constant. Indeed, this comes from the fact that, independently of its initial capital, the model allows the insurance company to sell one contract. However, as it is precised in section 2, in the case of insurance companies, the codes of law impose that, at any time, these companies should be able to justify enough resources to cover the obligations contracted towards their clients. This condition imposes an upper limit for the number of contracts the company can have. In the work we present here, several contracts are considered. We obtain a stochastic differential equation with respect to a random measure and introduce the utility for the shareholders as in Mnif, Sulem (2005) to be the maximized discounted flow of dividends. We prove that the value function is regular enough (enjoys the Lipschitz property) and satisfies the associated Hamilton-Jacobi-Bellman Variational Inequality in the viscosity sense. We also provide an uniqueness result for the viscosity solution in the class of continuous functions of at most linear growth. We emphasize that the limitation of the number of contracts which comes from the codes of insurance, allows us to get the Lipschitz property of the value function V. This property insures that an initial capital close to 0 will induce a zero-expected gain (unlike the collective risk model). Moreover, in this case, the dynamic programming principle follows in a standard way, while it was only assumed by the authors of [9].

The paper is organized as follows. In the first section we present a simple example showing the limits of the collective risk model. The second section is concerned with the insurance problem with several contracts. We introduce the model, the basic assumptions and prove some elementary properties of the value function V. In the third section, we show that the value function is a viscosity solution of the associated Hamilton-Jacobi-Bellman variational inequality. The fourth section provides a comparison result which allows to obtain the uniqueness of the viscosity solution for the given variational inequality. A numerical example is given in the last section.

2 The limits of the collective risk model. A counter example

We consider the following special case of the collective risk model introduced by Mnif, Sulem (2005). We assume that the claims are generated by a Poisson process N with intensity 1 on a complete probability space $(\Omega, \mathcal{F}, P)$. We denote by $(\mathcal{F}_t)_{t\geq 0}$ the filtration generated by the random measure associated to N, completed by the family of P-null sets. Given an $\mathcal{F}_t-$adapted process

$\alpha_t \in [0,1]$ (retention level), the premium rate is

$$p(\alpha_t) = k_1 - k_2 + (1 + k_2)\,\alpha_t, \text{ for all } t \geq 0,$$

where $0 \leq k_1 \leq k_2$ are proportional factors. Moreover, if L denotes the F_t-adapted process of cumulative dividends, then the reserve of the insurance company satisfies the equation

$$X_t^{x,\alpha,L} = x + \int_0^t p(\alpha_s)ds - N_t - \int_0^t dL_u.$$

The process L should be right-continuous, non-decreasing and such that $L_{0-} = 0$ and $L_t - L_{t-} \leq X_{t-}^{x,u,L}$ for all $t \geq 0$. We introduce the first jump time for the Poisson process N

$$\tau_1 = \inf\{t \geq 0 : N_t = 1\}.$$

Obviously, τ_1 is of exponential law with intensity 1, and, in particular,

$$P(\tau_1 > 1) = e^{-1}.$$

If we consider the strategy (α, L) given by

$$\begin{cases} \alpha \equiv 1, \\ L_t(\omega) = I_{\{\tau_1 > 1\}}(\omega) I_{\{t \geq 1\}}(t), \end{cases}$$

then (α, L) is admissible and the ruin time

$$\tau^{x,\alpha,L} > 1 \text{ on } \{\tau_1 > 1\}.$$

Indeed,

$$X_t^{x,\alpha,L} = x + (1 + k_1)\,t - N_t - \int_0^t dL_u,$$

and on $\{\tau_1 > 1\}$ we have that

$$X_t^{x,\alpha,L} = x + (1 + k_1)\,t,$$

for all $t < 1$.

It follows that

$$V(x) \geq E\left[\int_0^{\tau^{x,\alpha,L}} e^{-rt} dL_t\right] \geq E\left[e^{-r} I_{\{\tau_1 > 1\}}\right] \geq e^{-(r+1)},$$

for all $x > 0$. Obviously

$$V(0+) \geq e^{-(r+1)} > 0.$$

Therefore, investing an arbitrarily small capital in the insurance company, we expect to gain more than $e^{-(r+1)}$. This contradicts theorem 3.3 in [9]. This problem is due mainly to the fact that, independent of the initial capital, the insurance company is allowed to hold one contract.

However, the insurance law requires that, at any moment, the companies should be able to cover any liabilities that have been incurred on insurance contracts as far as can be reasonably foreseen. Experience of similar claim development trends is of particular relevance. Usually, the solvency margin is computed with respect to both the premium rates and the average claim. According to the current Solvency I prudence regime, "the life insurance capital requirements are arrived at by multiplying a factor of 4% to the mathematical reserves of participating business (for unit-linked business the factor is reduced to 1%) plus a factor of 0.3% to the sum-at-risk" (CEA and Mercer Oliver Wyman, *Solvency Assessment Models Compared,* http://www.cea.assur.org/cea/download/publ/article221.pdf).

The suitable formulae should take into account the specificities of life, non-life and reinsurance business. Various methods are, therefore, available. To give an example, according to the French legislation (Code des Assurances, R334-13) for the life insurance, the solvency margin (to be replaced by the Solvency Capital Requirement for Solvency II) should be superior to the result obtained by multiplying 0,3% of the capital under risk with the ratio between the capital under risk after reinsurance and the capital under risk before reinsurance computed for the previous exercise. The latter ratio cannot be inferior to 50%. To keep it simple, at time t the result obtained by multiplying a constant ζ_0 (depending on previous experience and the type of insurance business) by the average claim per contract and by the number of contracts n_t should not exceed the fortune of the insurance company:

$$\zeta_0 \times n_t \times \text{average claim} \leq \text{fortune at time } t. \tag{1}$$

Corroborating these elements, it appears obvious that the simple collective risk model should be improved to a model involving several contracts. We emphasize the fact that only quantitative requirements are taken into consideration (therefore, the model covers only part of Solvency II Pillar 1 requirements).

3 The insurance problem with several contracts

We introduce a complete probability space $(\Omega, \mathcal{F}, P)$. In order to model the claims, as for Mnif, Sulem (2005), we use a compound Poisson process given by a random measure $\mu(dtdy)$ on $\mathbb{R}_+ \times B$, with $B \subset \mathbb{R}_+ \setminus \{0\}$. Moreover, we assume that the compensator of μ takes the form $dt\pi(dy)$ and that the measure π is finite $\pi(dy) = \beta G(dy)$ for some probability measure $G(dy)$ on B and some positive constant β.

Throughout the section, we let Y denote a generic random variable distributed according to $G(dy)$.

We consider the natural filtration $(\mathcal{F}_t)_{t\geq 0}$ generated by the random measure μ. We call retention level any $(\mathcal{F}_t)$-adapted process $(u_t)_{t\geq 0}$ which specifies that, given a claim y at time $t \geq 0$, the direct insurer covers $y \wedge u_t$ while the reinsurance company covers the excess of loss $(y - u_t)^+$.

Since we are going to consider several insurance contracts, we introduce a function f depending both on the number of insurance contracts and on the risk taken by the company to model the claims $f : \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}_+$. If the company chooses some retention level u_t, then the actual premium rate per contract is given as in Asmussenet al. (2000), or, again, in Mnif, Sulem (2005)

$$p(u_t) = (1 + k_1)\beta\nu - (1 + k_2)\beta E\left[f(1, (Y - u_t)^+)\right] \text{ for all } t \geq 0, \tag{2}$$

where k_i are real constants satisfying $0 \leq k_1 < k_2$ and

$$\nu = \int_B f(1, y)G(dy) = E[f(1, Y)]. \tag{3}$$

The first term in (2) is the premium received from the client, while the second term is the quantity paid to the reinsurer.

Given the initial fortune $x \geq 0$ and the retention level u, if L stands for the $(\mathcal{F}_t)-$adapted process representing the cumulative dividends paid up to the time t, n_t denotes the number of contracts of the insurance company at time t, and $X_t^{x,u,L}$ the fortune of the company , then we must have

$$X_t^{x,u,L} = x + \int_0^t n_s p(u_s)ds - \int_0^{t+} \int_B f(n_s, y \wedge u_s)\mu(dsdy) - \int_0^t dL_s. \tag{4}$$

If we denote by a the quantity

$$a = \frac{1}{\zeta_0 \nu},$$

then, from (1) we get that the maximum number of insurance contracts is $n_t^{\max} = aX_t^{x,u,L}$. We have the following equation

$$X_t^{x,u,L} = x + a\int_0^t X_s^{x,u,L} p(u_s)ds - \int_0^{t+} \int_B f(aX_{s-}^{x,u,L}, y \wedge u_s)\mu(dsdy) - \int_0^t dL_s, \tag{5}$$

and introduce the cost functional

$$J(x,u,L) = E\left[\int_0^\tau e^{-rs} dL_s\right], \tag{6}$$

where r is some discount factor and τ is the ruin time

$$\tau = \inf\left\{t \geq 0 : X_t^{x,u,L} \leq 0\right\}.$$

Our value function V will be defined as the maximum over some family of admissible couples (u, L) of the cost functional J.

In practice, whenever the solvency condition is not satisfied, one of the following two events may occur. In the first case, a capital infusion from the shareholders intervenes. In the second one, an external referee solves the problem: either by transferring some of the contracts to other insurance companies, or by dissolving the contracts in final phase. The Solvency II framework states that as soon as the Solvency Capital Requirement (SCR) is not satisfied, supervisory action will be triggered. However, if the Minimum Capital Requirement (MCR) is not satisfied, the control authority can invoke severe measures (including closure of the company). From the mathematical point of view, we do not allow capital infusions, these being obtained by taking a larger initial reserve. On the contrary, the latter events may appear and they allow the variation of the number of contracts.

Let us now return to the function f modelling the claims. It is natural to suppose that the claims increase with the number of contracts and are null if the company has no contract. Moreover, the claims should increase with the risks covered and should be 0 if dealing with no risk. If the number of contracts is positive and the risk covered by these contracts is not null, then the claims are expected to be strictly positive. An utility function is usually supposed to be concave. If we are given a concave function v such that $v(0) = 0$, then

$$v(\lambda x) \geq \lambda v(x),$$

for any $\lambda \leq 1$. Since any nonlinearity in (5) may only come from f, in order to obtain the previous property for our utility function V, one should assume that f is convex in the first variable. These assumptions give

Assumption 1 ***(A1)*** *Suppose that the function $f : \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}_+$ satisfies:*

- $f(\cdot, y)$ is convex, non decreasing and $f(0, y) = 0$ for all $y \in \mathbb{R}_+$;

- $f(x, \cdot)$ is increasing and $f(x, 0) = 0$;

- $f(x, y) > 0$ if $x > 0$ and $y > 0$;

- f is uniformly continuous on $\mathbb{R}_+ \times \mathbb{R}$;

- *$f(x,y)$ is Lipschitz in x, uniformly in $y \in \mathbb{R}_+$.*

One expects to cover expenditures through the premium received

$$p(u_t) \geq \beta E[f(1, Y \wedge u_t)].$$

Recall that $p(0) - \beta E[f(1,0)] < 0$ and that $lim_{u\to\infty}(p(u) - \beta E[f(1, Y \wedge u)]) > 0$ (recall the definitions (2) and (3) of p and ν, respectively) and we obtain the existence of some $\underline{u} > 0$ such that

$$p(u) \geq \beta E[f(1, Y \wedge u)], \tag{7}$$

for all $u \geq \underline{u}$. Thus, we are going to consider only the retention levels u_t satisfying

$$u_t \geq \underline{u}. \tag{8}$$

One should impose that the dividends paid at some time t do not exceed the reserve at the same time. Therefore, we call admissible strategy the couple of $(\mathcal{F}_t)-$adapted processes (u, L) such that u satisfies (8) and L is càdlàg, non decreasing, $L_{0-} = 0$ and $L_t - L_{t-} \leq X_{t-}^{x,u,L}$ for almost every (t, ω). We should first prove the existence of such admissible strategies.

Remark 2 *If l is an $(\mathcal{F}_t)-$adapted processes which is càdlàg, non decreasing, $l_{0-} = 0$, then, for any initial condition $x \geq 0$, and any $(\mathcal{F}_t)-$adapted processes u which satisfies (8), there exists a unique $\mathcal{F}_t-$adapted right-continuous process $X_t^{x,\alpha,l}$ with left-hand limits which satisfies the equation*

$$X_t^{x,u,l} = x + a\int_0^t X_s^{x,u,l} p(u_s)ds - \int_0^{t+}\int_B f(aX_{s-}^{x,u,l}, y \wedge u_s)\mu(dsdy) - \int_0^t dl_s. \tag{9}$$

(see also Ikeda, Watanabe (1989) IV, Theorem 9.1). We define the ruin time $\tau = \inf\left\{t \geq 0 : X_t^{x,u,l} \leq 0\right\}$. Obviously, on $\{t < \tau\}$ we have $\triangle l_t = l_t - l_{t-} \leq X_{t-}^{x,u,l}$. Let us define the process

$$L_t = l_t 1_{\{t<\tau\}} + \left(\triangle l_t \wedge X_{t-}^{x,u,l}\right) 1_{\{t=\tau\}}.$$

We get an $(\mathcal{F}_t)-$adapted process which is càdlàg, non decreasing, and $L_{0-} = 0$. Let $X^{x,u,L}$ denote the solution of (9) with L instead of l. We notice that (u, L) is admissible in the sense that $L_t - L_{t-} \leq X_{t-}^{x,u,L}$ for almost every (t, ω).

For all initial reserve $x \geq 0$, we denote by $\mathcal{A}(x)$ the set of admissible strategies described above. The value function is defined by

$$V(x) = \sup_{(u,L)\in\mathcal{A}(x)} J(x, u, L).$$

Proposition 3 *(Comparison for solutions of (9)) Given two $(\mathcal{F}_t)-$adapted processes u and l such that u satisfies (8) and l is càdlàg, non decreasing, and*

$l_{0-} = 0$, and two initial states $0 \leq x \leq x'$, the solutions of (9) $X^{x,u,l}$ and $X^{x',u,l}$ starting from x (respectively x') and associated with the pair (u, l) satisfy

$$X_t^{x,u,l} \leq X_t^{x',u,l}, \text{ for all } t, \ P - a.s.$$

PROOF. Let us consider the sequence of functions $\phi_n \in C^1(\mathbb{R})$ such that $\phi_n(x'') = 0$ for all $x'' \leq 0$, $0 \leq \phi_n'(x'') \leq 1$, for all $x'' \in \mathbb{R}$, and $\phi_n(x'') \uparrow (x'')^+$ as $n \to \infty$. A simple application of Itô's formula yields

$$\phi_n\left(X_t^{x,u,l} - X_t^{x',u,l}\right) = I_1 + I_2, \tag{10}$$

where

$$I_1 = \int_0^t ap(u_s)\left(X_s^{x,u,l} - X_s^{x',u,l}\right)\phi_n'\left(X_s^{x,u,l} - X_s^{x',u,l}\right)ds,$$

$$I_2 =$$

$$\int_0^{t+}\int_B \phi_n\left(X_{s-}^{x,u,l} - X_{s-}^{x',u,l} - f\left(aX_{s-}^{x,u,l}, y \wedge u_s\right) + f\left(aX_{s-}^{x',u,l}, y \wedge u_s\right)\right)\mu(dsdy)$$

$$-\int_0^{t+}\int_B \phi_n\left(X_{s-}^{x,u,l} - X_{s-}^{x',u,l}\right)\mu(dsdy).$$

It is obvious that

$$I_1 \leq C\int_0^t \left(X_s^{x,u,l} - X_s^{x',u,l}\right)^+ ds,$$

where C is a constant independent of x and x'. Since a can be chosen arbitrarily small (for that, it is enough to recall $a = \frac{1}{\zeta_0 \nu}$ and then choose an arbitrarily small monetary unit such that the quantity ν becomes large), we may assume that $aK_0 \leq 1$ (here K_0 denotes the Lipschitz constant for f). Then the function $x \longmapsto x - f(ax, y)$ is increasing for all $y \in \mathbb{R}_+$. Therefore, we get

$$I_2 \leq 0.$$

Combining the two estimates for I_1 and I_2, we have

$$E\left[\phi_n\left(X_t^{x,u,l} - X_t^{x',u,l}\right)\right] \leq C\int_0^t E\left[\left(X_s^{x,u,l} - X_s^{x',u,l}\right)^+\right]ds.$$

We allow $n \to \infty$ to obtain

$$E\left[\left(X_t^{x,u,l} - X_t^{x',u,l}\right)^+\right] \leq C\int_0^t E\left[\left(X_s^{x,u,l} - X_s^{x',u,l}\right)^+\right]ds.$$

Finally, Gronwall's inequality yields

$$E\left[\left(X_t^{x,u,l} - X_t^{x',u,l}\right)^+\right] = 0.$$

The proof of our Proposition is complete.

If the initial fortune is fixed, then the company has to make a choice over some family of admissible strategies. One may naturally wonder whether the same strategies are valid when dealing with a greater initial reserve or not. The answer is affirmative as proven by the following Proposition.

Proposition 4 *If $0 \leq x \leq x'$ are two initial capitals and if (u, L) is an admissible strategy for x, then (u, L) is also admissible for x'.*

PROOF. Indeed, if $X_t^{x,u,L}$ (respectively $X_t^{x',u,L}$) denote the solutions of (9) starting from x (respectively x') associated with the control pair (u, L), then the comparison result yields

$$X_t^{x,u,L} \leq X_t^{x',u,L}, dtdP - a.e. \text{ on } [0,\infty) \times \Omega.$$

Now, since L is admissible for x, we have

$$L_t - L_{t-} \leq X_{t-}^{x,u,L} \leq X_{t-}^{x',u,L}, dtdP - a.e.,$$

and L is again admissible for x'. Moreover, if τ denotes the ruin time for $X_t^{x,u,L}$ and τ' denotes the ruin time for $X_t^{x',u,L}$, then, obviously

$$\tau \leq \tau', \ P - a.s.$$

As one expects, using the previous results, we find that the utility function of the insurance company increases with the initial reserve. Since our strategy involves a dynamic programming approach, we would like to have finite value function. We suppose that the following assumption holds true

Assumption 5 *(A2) The discount factor r in (6) satisfies*

$$r > \frac{2(1+k_1)\beta}{\zeta_0}$$

Given an economic framework in which the discount factor r is fixed, the above assumption says that the time between two claims is great enough to justify the demand for small solvency translated in the small constant ζ_0).

Under this Assumption, we provide an upper bound estimate as well as Lipschitz regularity of the value function.

Proposition 6 *The value function V is non decreasing, enjoys the Lipschitz property and satisfies*

$$V(x) \leq Kx, \tag{11}$$

for some large enough positive constant K.

PROOF. The first assertion is straightforward from the previous Proposition. In order to establish the upper bound (11), we notice that

$$X_t^{x,u,L} \leq x + \frac{(1+k_1)\beta}{\zeta_0}\int_0^t X_s^{x,u,L} ds,$$

for all $t \geq 0$. Gronwall's inequality yields

$$X_t^{x,u,L} \leq x e^{\frac{(1+k_1)\beta}{\zeta_0}t}. \tag{12}$$

We write Itô's formula for $e^{-rt}X_t^{x,u,L}$ and use (12) together with **(A2)** to obtain

$$J(x,u,L) \leq Cx.$$

Here C is a constant which may change from line to line. Let us fix $x, x' \geq 0$. Suppose that $(u,L) \in \mathcal{A}(x+x')$ and notice that, in this case, $\left(u, \frac{x}{x+x'}L\right) \in \mathcal{A}(x)$. Indeed,

$$\begin{aligned} X_t^{x+x',u,L} &= (x+x') + a\int_0^t X_s^{x+x',u,L} p(u_s) ds \\ &\quad - \int_0^{t+}\int_B f(aX_{s-}^{x+x',u,L}, y \wedge u_s)\mu(dsdy) - \int_0^t dL_s, \end{aligned}$$

and, by multiplying the latter equality by $\frac{x}{x+x'}$, we get

$$\begin{aligned} \frac{x}{x+x'}X_t^{x+x',u,L} &= x + \int_0^t ap(u_s)\frac{x}{x+x'}X_s^{x+x',u,L} ds \\ &\quad - \int_0^{t+}\int_B \frac{x}{x+x'} f\left(aX_{s-}^{x+x',u,L}, y \wedge u_s\right)\mu(dsdy) \\ &\quad - \int_0^t d\left(\frac{x}{x+x'}L_s\right). \end{aligned}$$

On the other hand,

$$\begin{aligned} X_t^{x,u,\frac{x}{x+x'}L} &= x + \int_0^t ap(u_s)X_s^{x,u,\frac{x}{x+x'}L} ds \\ &\quad - \int_0^{t+}\int_B f\left(aX_{s-}^{x',u,\frac{x}{x+x'}L}, y \wedge u_s\right)\mu(dsdy) \\ &\quad - \int_0^t d\left(\frac{x}{x+x'}L_s\right). \end{aligned}$$

Now, let the functions $\phi_n \in C^1(\mathbb{R})$ be such that $\phi_n(x'') = 0$ for all $x'' \leq 0$, and $0 \leq \phi_n'(x'') \leq 1$, for all $x'' \in \mathbb{R}$, and $\phi_n(x'') \uparrow (x'')^+$ as $n \to \infty$. We make the following notation

$$\frac{x}{x+x'}X_\cdot^{x+x',u,L} = Y_\cdot.$$

We apply Itô's formula to have

$$\phi_n\left(Y_t - X_t^{x,u,\frac{x}{x+x'}L}\right) = I_1 + I_2, \tag{13}$$

where

$$I_1 = \int_0^t ap(u_s)\left(Y_s - X_s^{x,u,\frac{x}{x+x'}L}\right)\phi_n'\left(Y_s - X_s^{x,u,\frac{x}{x+x'}L}\right)ds,$$

$$I_2 =$$

$$\int_0^{t+}\int_B \phi_n\left(Y_s - X_s^{x,u,\frac{x}{x+x'}L} - \frac{x}{x+x'}f\left(aX_s^{x+x',u,L}, y\wedge u_s\right) + f\left(aX_s^{x,u,\frac{x}{x+x'}L}, y\wedge u_s\right)\right)\mu(dsdy)$$

$$-\int_0^{t+}\int_B \phi_n\left(Y_s - X_s^{x,u,\frac{x}{x+x'}L}\right)\mu(dsdy).$$

It is obvious that

$$I_1 \le C\int_0^t \left(Y_s - X_s^{x,u,\frac{x}{x+x'}L}\right)^+ ds,$$

where C is a constant independent of x and x', and we use the convexity of f in the first variable and $f(0,\cdot) = 0$, together with the monotonicity of ϕ_n to get (as in the proof of the comparison result),

$$I_2 \le 0.$$

Thus we obtain, as in the comparison result,

$$\frac{x}{x+x'}X_t^{x+x',u,L} \le X_t^{x,u,\frac{x}{x+x'}L} \quad dtdP - a.e. \text{ on } [0,\infty)\times\Omega.$$

Obviously, $\left(u, \frac{x}{x+x'}L\right)$ is an admissible strategy for the initial reserve x. If τ is the ruin time for the strategy (u, L) for the initial reserve $x + x'$, then the above inequality states that the ruin time for the strategy $\left(u, \frac{x}{x+x'}L\right)$ when the initial reserve is x is greater than or equal to τ. Therefore, we have

$$V(x+x') = \frac{x+x'}{x}\sup_{(u,L)\in\mathcal{A}(x+x')} E\left[\int_0^\tau e^{-rs}d\left(\frac{x}{x+x'}L_s\right)\right] \le \frac{x+x'}{x}V(x).$$

and (11) gives the Lipschitz property of V. The proof of the Proposition is complete.

4 Hamilton Jacobi Bellman Variational Inequality

We have already seen that our value function V is increasing and Lipschitz continuous. These properties allow us to prove in a standard way that V satisfies the following **Dynamic Programming Principle**

Principle 7 ***(DPP)***

$$V(x) = \sup_{(u,L)\in\mathcal{A}(x)} E\left[e^{-r(t\wedge\tau)}V(X^{x,u,L}_{t\wedge\tau}) + \int_0^{t\wedge\tau} e^{-rs}dL_s\right],$$

for all $t \geq 0,\ x \geq 0$.

For further literature on the subject, the reader to referred to Fleming, Soner (1993), Krylov (1980), or Yong, Zhou (1999) (theorem 4.3.3), for diffusion state processes or to Pham (1998) in the case of jump diffusion processes.

We consider at this point the following HJB variational inequality:

$$\begin{cases} \max\{H(x,V,V'(x)), 1-V'(x)\} = 0 \text{ in } \mathbb{R}_+^*, \\ V(0) = 0. \end{cases}, \tag{14}$$

where

$$\begin{aligned} &H(x,V,q) \\ =&\sup_{u\geq\underline{u}}\left\{-rV(x) + axp(u)q + \int_B \left[V(x - f(ax, y\wedge u)) - V(x)\right]\pi(dy)\right\}. \end{aligned} \tag{15}$$

Let us recall that $C^{1,1}(\mathbb{R}_+)$ stands for the class of all real-valued, differentiable functions on $\mathbb{R}_+$ such that the derivative is locally Lipschitz.

We also recall the definition of the viscosity supersolution, respectively viscosity subsolution.

Definition 8 *(i) Any lower semi-continuous (respectively upper semi-continuous) function v is a viscosity supersolution (subsolution) of (14) if $v(0) \geq 0$ (≤ 0) and*

$$\max\{H(x,\varphi,\varphi'(x)), 1-\varphi'(x)\} \leq 0,$$

(respectively ≥ 0) whenever $\varphi \in C^{1,1}(\mathbb{R}_+)$ is such that $v-\varphi$ has a global minimum (maximum) at $x > 0$.

(ii) A function v is a viscosity solution of (14) if it is both super and subsolution.

Theorem 9 *The value function V is a viscosity solution for the associated Hamilton-Jacobi-Bellman Variational Inequality (14).*

PROOF. First, we prove that V is a viscosity supersolution for (14). In order to do this, let us consider $x \in \mathbb{R}_+^*$ and a $C^{1,1}$ test function φ such that

$V(x')-\varphi(x') \geq V(x)-\varphi(x) = 0$, for all $x' \in \mathbb{R}_+^*$. Moreover, consider $0 < h < x$ and the admissible strategy $(u, L) \in \mathcal{A}(x)$ where $L_s = h$, for all $s \geq 0$ and u is admissible and arbitrarily chosen. We have

$$\begin{aligned}\varphi(x) = V(x) &\geq E\left[\int_0^{t\wedge\tau} e^{-rs} dL_s + e^{-r(t\wedge\tau)} V(X_{t\wedge\tau}^{x,u,L})\right] \\ &\geq h + E\left[e^{-r(t\wedge\tau)}\varphi(X_{t\wedge\tau}^{x,u,L})\right],\end{aligned}$$

for all $t \geq 0$. We take the limit as $t \to 0+$ and get

$$\varphi(x) \geq h + \varphi(x-h).$$

This latter inequality yields

$$1 - \varphi'(x) \leq 0. \tag{16}$$

In order to prove $H(x, \varphi, \varphi'(x)) \leq 0$, we consider the admissible pair $L_s = 0$, $u_s = u_0$, for all $s \geq 0$ (here $u_0 \geq \underline{u}$ is arbitrarily chosen). We apply Itô's formula to $e^{-r(t\wedge\tau)}\varphi(X_{t\wedge\tau}^{x,u,L})$ to obtain

$$\begin{aligned}&E\left[e^{-r(t\wedge\tau)}\varphi(X_{t\wedge\tau}^{x,u,L})\right] - \varphi(x) \\ &= E\left[\int_0^{t\wedge\tau}\left(-re^{-rs}\varphi(X_s^{x,u,L}) + e^{-rs} a X_s^{x,u,L} p(u_0)\varphi'(X_s^{x,u,L})\right) ds\right] \\ &+ E\left[\int_0^{t\wedge\tau}\int_B e^{-rs}\left(\varphi\left(X_{s-}^{x,u,L} - f\left(aX_{s-}^{x,u,L}, y\wedge u_0\right)\right) - \varphi\left(X_{s-}^{x,u,L}\right)\right)\mu(dsdy)\right].\end{aligned}$$

Recalling that $\varphi(x) \geq E\left[e^{-r(t\wedge\tau)}V(X_{t\wedge\tau}^{x,u,L})\right]$, and dividing by $t > 0$, we have

$$\begin{aligned}0 &\geq E\left[\frac{1}{t}\int_0^{t\wedge\tau}\left(-re^{-rs}\varphi(X_s^{x,u,L}) + e^{-rs} a X_s^{x,u,L} p(u_0)\varphi'(X_s^{x,u,L})\right) ds\right] \\ &+ E\left[\frac{1}{t}\int_0^{t\wedge\tau}\int_B e^{-rs}\left(\varphi\left(X_{s-}^{x,u,L} - f\left(aX_{s-}^{x,u,L}, y\wedge u_0\right)\right) - \varphi\left(X_{s-}^{x,u,L}\right)\right)\mu(dsdy)\right] \\ &\geq E\left[\frac{1}{t}\int_0^{t\wedge\tau}\left(-r\varphi(x) + e^{-rt} axp(u_0)\varphi'(x)\right) ds\right] \\ &+ E\left[\frac{1}{t}\int_0^{t\wedge\tau} ds \int_B \left(e^{-rt}\varphi\left(x - f\left(ax, y\wedge u_0\right)\right) - \varphi\left(x\right)\right)\pi(dy)\right] \\ &- O\left(E\left[\sup_{s\leq t\wedge\tau} e^{-rs}\left|X_s^{x,u,L} - x\right|\right]\right),\end{aligned} \tag{17}$$

where $O(\delta) \to 0$ whenever $\delta \to 0$.

We wish to prove that $E\left[\sup_{s\leq t\wedge\tau} e^{-rs}\left|X_s^{x,u,L} - x\right|\right] \to 0$, when $t \to 0$. In order to do this, we use

$$
\begin{aligned}
\left|X_s^{x,u,L} - x\right| \leq & \int_0^s ap(u_0) X_{s'}^{x,u,L} ds' \\
& + \int_0^{s+} \int_B f(aX_{s'-}^{x,u,L}, y \wedge u_0) \mu(ds'dy).
\end{aligned}
$$

Therefore, with the notation $C_0 = \frac{(1+k_1)\beta}{\zeta_0}$, we have, for some constant C,

$$
\begin{aligned}
\left|X_s^{x,u,L} - x\right| \leq & \, x\left(e^{C_0 s} - 1\right) \\
& + C \int_0^{s+} \int_B e^{C_0 s'} \mu(ds'dy)
\end{aligned}
$$

for all $0 \leq s \leq t \wedge \tau$ (we use the Lipschitz property of f in x uniformly in y, $f(0, \cdot) = 0$ and the upper bound for $X_{s'}^{x,u,L}$ given by (12)) . We multiply the last inequality by e^{-rs}, take the supremum over all $0 \leq s \leq t \wedge \tau$, then the expectation with respect to P to obtain

$$
\lim_{t \to 0+} E\left[\sup_{s \leq t \wedge \tau} e^{-rs} \left|X_s^{x,u,L} - x\right|\right] = 0. \tag{18}
$$

Notice that

$$
\frac{E[t \wedge \tau]}{t} \geq 1 - P(\tau \leq t) \geq 1 - P(\eta_1 \leq t),
$$

where η_1 is the first time a claim occurs (it follows the exponential law). Consequently,

$$
\lim_{t \to 0+} \frac{E[t \wedge \tau]}{t} = 1. \tag{19}
$$

Returning to (17) we let $t \to 0+$ and use (18) and (19) to get

$$
\begin{aligned}
0 \geq & \left(-r\varphi(x) + axp(u_0)\varphi'(x)\right) \\
& + \int_B \left\{\varphi\left(x - f\left(ax, y \wedge u_0\right)\right) - \varphi\left(x\right)\right\} \pi(dy)
\end{aligned} \tag{20}
$$

Combining (20) and (16), we prove that V is a viscosity supersolution for (14).

In order to prove that the value function is a viscosity subsolution for (14), we fix $x > 0$ and consider an arbitrary test function $\varphi \in C^{1,1}$ such that $V(x') - \varphi(x') \leq V(x) - \varphi(x) = 0$, for all $x' \in \mathbb{R}_+$. Let us suppose that the subsolution inequality does not hold. Therefore, there exists $\delta > 0$ such that

$$
\max\left\{H(x, \varphi, \varphi'(x)), 1 - \varphi'(x)\right\} < -\delta.
$$

We use the continuity of H and of φ' to obtain the existence of some $\eta \in \left(0, x \wedge \frac{\delta}{4K_\varphi}\right)$, where K_φ denotes the Lipschitz constant for φ on $[0, e^r x]$, such that

$$
\max\left\{H(x', \varphi, \varphi'(x')), 1 - \varphi'(x')\right\} < -\delta, \; if \; x' \in B(x, \eta). \tag{21}
$$

Let us consider an arbitrary strategy $(u, L) \in \mathcal{A}(x)$ and let $X^{x,u,L}$ denote the solution of (9) for (u, L) instead of (u, l). We define the stopping time

$$\sigma = \inf\{t \geq 0 : \ X_t^{x,u,L} \notin B(x, \eta)\}.$$

Obviously $\sigma \leq \tau$ (the ruin time). We apply Itô's formula to $e^{-r(t\wedge\sigma)}\varphi(X_{t\wedge\sigma}^{x,u,L})$ and write

$$\begin{aligned}
&E\left[e^{-r(t\wedge\sigma)}\varphi(X_{t\wedge\sigma}^{x,u,L})\right] - \varphi(x) = \qquad (22)\\
&E\left[\int_0^{t\wedge\sigma}\left(-re^{-rs}\varphi(X_s^{x,u,L}) + e^{-rs}aX_s^{x,u,L}p(u_0)\varphi'(X_s^{x,u,L})\right)ds\right]\\
&+E\left[\int_0^{t\wedge\sigma}\int_B e^{-rs}\left(\varphi\left(X_{s-}^{x,u,L} - f\left(aX_{s-}^{x,u,L}, y\wedge u_0\right)\right) - \varphi\left(X_{s-}^{x,u,L}\right)\right)\mu(dsdy)\right]\\
&-E\left[\int_0^{t\wedge\sigma} e^{-rs}\varphi'\left(X_s^{x,u,L}\right)dL_s^c\right]\\
&+E\left[\sum_{s\leq t\wedge\sigma} e^{-rs}\left(\varphi\left(X_{s-}^{x,u,L} - \triangle L_s\right) - \varphi\left(X_{s-}^{x,u,L}\right)\right)\right].
\end{aligned}$$

For $s < t \wedge \sigma$ we have, from (21)

$$\begin{aligned}
&-re^{-rs}\varphi(X_s^{x,u,L}) + e^{-rs}aX_s^{x,u,L}p(u_0)\varphi'(X_s^{x,u,L}) \qquad (23)\\
&+e^{-rs}\int_B\left(\varphi\left(X_s^{x,u,L} - f\left(aX_s^{x,u,L}, y\wedge u_0\right)\right) - \varphi\left(X_s^{x,u,L}\right)\right)\pi(dy) < -\delta e^{-rs},
\end{aligned}$$

and, again from (21),

$$\varphi'\left(X_s^{x,u,L}\right) > 1.$$

It follows that

$$\varphi\left(X_{s-}^{x,u,L} - \triangle L_s\right) - \varphi\left(X_{s-}^{x,u,L}\right) \leq -\triangle L_s, \qquad (24)$$

and

$$\begin{aligned}
&\varphi\left(X_{s-}^{x,u,L} - f\left(aX_{s-}^{x,u,L}, y\wedge u_0\right)\right) - \varphi\left(X_{s-}^{x,u,L}\right)\\
\leq\,&\varphi\left(X_s^{x,u,L} - f\left(aX_s^{x,u,L}, y\wedge u_0\right)\right) - \varphi\left(X_s^{x,u,L}\right)\\
&+4K_\varphi\eta. \qquad (25)
\end{aligned}$$

We return to (22) and use (23), (24) and (25) to get

$$\begin{aligned}
E\left[e^{-r(t\wedge\sigma)}\varphi(X_{t\wedge\sigma}^{x,u,L})\right] - \varphi(x) \leq\,& \delta E\left[\frac{e^{-r(t\wedge\sigma)} - 1}{r}\right] + 4K_\varphi\eta E\left[\frac{1 - e^{-r(t\wedge\sigma)}}{r}\right]\\
&-E\left[\int_0^{t\wedge\sigma} e^{-rs}dL_s\right],
\end{aligned}$$

and, from this,

$$
\begin{aligned}
V(x) &= \varphi(x) \\
&\geq E\left[e^{-r(t\wedge\sigma)}\varphi(X_{t\wedge\sigma}^{x,u,L}) + \int_0^{t\wedge\sigma} e^{-rs} dL_s\right] \\
&\quad + (\delta - 4K_\varphi\eta)\, E\left[\frac{1-e^{-r(t\wedge\sigma)}}{r}\right]. \\
&\geq E\left[e^{-r(t\wedge\sigma)}\varphi(X_{t\wedge\sigma}^{x,u,L}) + \int_0^{t\wedge\sigma} e^{-rs} dL_s\right] + \frac{\delta - 4K_\varphi\eta}{2} E[t\wedge\sigma],
\end{aligned}
\tag{26}
$$

for t small enough. We can suppose that x is a strict global maximum point. Then there exists $\lambda > 0$ such that

$$
\sup_{x' \notin B^\circ(x,\eta)} (V(x') - \varphi(x')) = -\lambda.
$$

We use (26) and write

$$
\begin{aligned}
V(x) &\geq E\left[e^{-r(t\wedge\sigma)}V(X_{t\wedge\sigma}^{x,u,L}) + \int_0^{t\wedge\sigma} e^{-rs} dL_s\right] \\
&+ \quad \lambda E\left[e^{-r(t\wedge\sigma)}1_{\sigma\leq t}\right] + \frac{\delta - 4K_\varphi\eta}{2} tP(\sigma > t) \\
&\geq E\left[e^{-r(t\wedge\sigma)}V(X_{t\wedge\sigma}^{x,u,L}) + \int_0^{t\wedge\sigma} e^{-rs} dL_s\right] \\
&+ \quad \left(\lambda e^{-rt}\right) \wedge \left(\frac{\delta - 4K_\varphi\eta}{2} t\right)
\end{aligned}
\tag{27}
$$

The dynamic programming principle yields

$$
V(x) \leq \sup_{(u,L)} E\left[e^{-r(t\wedge\sigma)}V(X_{t\wedge\sigma}^{x,u,L}) + \int_0^{t\wedge\sigma} e^{-rs} dL_s\right]. \tag{28}
$$

Therefore, by the choice of $\eta < \frac{\delta}{4K_\varphi}$ and $\lambda > 0$, (27) contradicts (28). This proves that V is a viscosity subsolution for (14). Our Theorem is now complete.

5 The Comparison Theorem

The following Lemma provides an equivalent definition for the notions of viscosity super and subsolutions.

Lemma 10 *(i) A continuous function U is a viscosity supersolution for (14) in $\mathbb{R}_+^*$ if and only if, $U(0) \geq 0$ and, for any $x \in \mathbb{R}_+^*$ and any test function*

$\varphi \in C^{1,1}$ such that $U-\varphi$ has a global strict minimum at x, we have

$$\max\left\{H(x,U,\varphi'(x)),1-\varphi'(x)\right\} \leq 0. \tag{29}$$

(ii) A continuous function U is a viscosity subsolution for (14) in $\mathbb{R}_+^*$ if and only if, $U(0) \leq 0$ and, for any $x \in \mathbb{R}_+^*$ and any test function $\varphi \in C^{1,1}$ such that $U-\varphi$ has a global strict maximum at x, we have

$$\max\left\{H(x,U,\varphi'(x)),1-\varphi'(x)\right\} \geq 0. \tag{30}$$

PROOF. We only prove the assertion for viscosity supersolution, the proof for subsolution being similar.

Suppose that (i) holds true. For any test function $\varphi \in C^{1,1}$such that $U(x) = \varphi(x)$ and $U-\varphi$ has a global minimum at x, and all $\delta > 0$, we define

$$\varphi_\delta(x') = \varphi\left(x'\right) - \delta\left|x'-x\right|^2, \text{ for all } x' \in \mathbb{R}_+^*.$$

Then $\varphi_\delta \in C^{1,1}$ and $U-\varphi_\delta$ has a global strict minimum at x. The assumption implies that

$$\max\left\{H(x,U,\varphi_\delta'(x)),1-\varphi_\delta'(x)\right\} \leq 0.$$

Obviously, $U(x')-U(x) > \varphi_\delta(x')-\varphi_\delta(x)$, for all $x' \in \mathbb{R}_+^* \smallsetminus \{x\}$. The definition of H, together with the last inequality, yields

$$\max\left\{H(x,\varphi_\delta,\varphi'(x)),1-\varphi'(x)\right\} \leq 0. \tag{31}$$

Moreover, again from the definition of H,

$$\begin{aligned} H(x,\varphi,\varphi'(x)) &\leq H(x,\varphi_\delta,\varphi'(x)) \\ &\quad + \sup_{u \geq \underline{u}}\left(\int_B \left(\varphi\left(x-f(ax,y\wedge u\right) - \varphi_\delta\left(x-f(ax,y\wedge u)\right)\pi\left(dy\right)\right)\right) \\ &\leq H(x,\varphi_\delta,\varphi'(x)) + C\delta, \end{aligned}$$

where C is a generic constant independent of δ. We get, using (31) then taking the limit as $\delta \searrow 0$,

$$\max\left\{H(x,\varphi,\varphi'(x)),1-\varphi'(x)\right\} \leq 0.$$

For the converse, consider an arbitrary test function $\varphi \in C^{1,1}$ and $x \in \mathbb{R}_+^*$ such that

$$0 = U(x) - \varphi(x) < U(x') - \varphi(x'),$$

for all $x' \in \mathbb{R}_+^* \setminus \{x\}$. For $\varepsilon > 0$ such that $\varepsilon < \frac{x}{4}$, we define

$$\delta_\varepsilon = \sup_{x' \in B(x,4\varepsilon)} \left(U(x') - \varphi(x')\right) > 0.$$

It is obvious that $\lim_{\varepsilon\to 0} \searrow \delta_\varepsilon = 0$. We introduce

$$\varphi_\varepsilon = (U - \varphi - \delta_\varepsilon)\, 1_{[0,x-2\varepsilon]} + (U(0) - \varphi(0) - \delta_\varepsilon)\, 1_{\mathbb{R}_-}.$$

We consider some sequence of mollifiers $\rho_n \in C_c^\infty(\mathbb{R};\mathbb{R}_+)$, $Supp\ \rho_n \subset B\left(0, \frac{1}{n}\right)$ and $\int_{\mathbb{R}} \rho_n(t)dt = 1$. Since $U - \varphi$ is continuous, the sequence $\{\rho_n * \varphi_\varepsilon\}_n$ converges uniformly on $[0, x-3\varepsilon]$ to φ_ε. Then there exists a subsequence (denoted by (ρ_ε)) such that $Supp\ \rho_\varepsilon \subset B(0,\varepsilon)$ and

$$U(x') - \varphi(x') - 2\delta_\varepsilon \le (\rho_\varepsilon * \varphi_\varepsilon)(x') < U(x') - \varphi(x'),$$

for all $0 \le x' \le x - 3\varepsilon$ and all $\varepsilon > 0$. Finally, we define the function

$$F_\varepsilon(x') = \varphi(x') + (\rho_\varepsilon * \varphi_\varepsilon)(x').$$

It is obvious that $F_\varepsilon \in C^{1,1}$ has the following properties:

$$\begin{cases} F_\varepsilon(x') = \varphi(x'), \text{ if } x' \ge x - \varepsilon, \\ U(x') - 2\delta_\varepsilon \le F_\varepsilon(x'), \text{ if } 0 \le x' \le x - 3\varepsilon, \\ F_\varepsilon(x') < U(x'), \text{ if } x' \ne x. \end{cases}$$

The assumptions give

$$\max\{H(x, F_\varepsilon, F_\varepsilon'(x)), 1 - F_\varepsilon'(x)\} \le 0.$$

Let us put

$$\begin{aligned} G(x') = \sup_{u \ge \underline{u}} \{ & -r\,(U(x') - F_\varepsilon(x')) + ap(u)x'\,(\varphi'(x') - F_\varepsilon'(x')) \\ & + \int_B (U(x' - f(ax', y\wedge u)) - F_\varepsilon(x' - f(ax', y \wedge u)))\,\pi(dy) \\ & - \int_B (U(x') - F_\varepsilon(x'))\,\pi(dy)\}, \end{aligned}$$

for any $x' \in \mathbb{R}_+^*$.Then

$$H(x, U, \varphi'(x)) - H(x, F_\varepsilon, F_\varepsilon'(x)) \le G(x), \tag{32}$$

where

$$\begin{aligned} G(x) \le \sup_{u \ge \underline{u}} \Big\{ & \int_B (U(x - f(ax, y \wedge u)) - F_\varepsilon(x - f(ax, y\wedge u)))\,\pi(dy) \\ & - \int_B (U(x) - F_\varepsilon(x))\,\pi(dy) \Big\}. \end{aligned}$$

We then consider the sets $B^u = \left\{ y \in B : x - f(ax, y \wedge u) \in \overline{B}(x, 3\varepsilon) \right\}$ and get

$$
\begin{aligned}
&\int_B \left(U(x - f(ax, y \wedge u)) - F_\varepsilon\left(x - f(ax, y \wedge u)\right)\right)\pi(dy) \\
&\leq \int_{B \setminus B^u} 2\delta_\varepsilon \pi(dy) + C\pi(B^u) \\
&\leq 2\beta\delta_\varepsilon + C\pi(B^u),
\end{aligned} \tag{33}
$$

where $C > 0$ is a generic constant independent of ε. Moreover, if $y \in B^u$, then

$$x - f(ax, y \wedge u) \geq x - 3\varepsilon.$$

Therefore,

$$f(ax, y \wedge \underline{u}) \leq f(ax, y \wedge u) \leq 3\varepsilon.$$

Since $f(ax, y \wedge \underline{u}) > 0$ for $xy > 0$ and $f(ax, \cdot)$ is nondecreasing, we deduce the existence of some $\eta_\varepsilon > 0$ such that $\eta_\varepsilon \to 0$ as $\varepsilon \to 0$ and $y \in B^u$ only if $y \leq \eta_\varepsilon$. Thus, returning to (33), we get

$$
\begin{aligned}
&\int_B \left(U(x - f(ax, y \wedge u)) - F_\varepsilon\left(x - f(ax, y \wedge u)\right)\right)\pi(dy) \\
&\leq C\delta_\varepsilon + C\pi(B \cap [0, \eta_\varepsilon]).
\end{aligned}
$$

Consequently,

$$G(x) \leq C\delta_\varepsilon + C\pi(B \cap [0, \eta_\varepsilon]). \tag{34}$$

Recall that $0 \notin B$. Thus, using (34) in (32) and taking the limit as $\varepsilon \to 0$, we obtain

$$H(x, U, \varphi') \leq 0,$$

and (i) follows.

The assertion (ii) follows in the same way.

Under the assumption **(A2)** we are able to prove the following result on the comparison of viscosity solutions for (14).

Theorem 11 *Let U and V be respectively a continuous viscosity subsolution and a continuous supersolution for (14) both of at most linear growth. Then, if **(A2)** holds true, we have*

$$U(x) \leq V(x), \text{ for all } x \in \mathbb{R}_+^*.$$

PROOF. For $\delta > 0$ and $\varepsilon > 0$, we denote by $\Phi_{\varepsilon,\delta}$ the function $\Phi_{\varepsilon,\delta} : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R} \cup \{-\infty\}$ given by

$$\Phi_{\varepsilon,\delta}(x, x') = U(x) - V(x') - \frac{1}{2\varepsilon}(x - x')^2 - \delta\left(x^2 + (x')^2\right), \tag{35}$$

for all $x, x' \geq 0$. Suppose that for some $x_0 \in \mathbb{R}_+^*$ and some $\theta > 0$ we have

$$U(x_0) - V(x_0) \geq \theta.$$

Since $\Phi_{\varepsilon,\delta}$ is upper semi-continuous and U and V are of linear growth, there exists a global maximum point of $\Phi_{\varepsilon,\delta}$, denoted by $(x_{\varepsilon,\delta}, x'_{\varepsilon,\delta}) \in \mathbb{R}_+ \times \mathbb{R}_+$. Obviously, since $\Phi_{\varepsilon,\delta}(0, x') \leq 0$ for all $x' \in \mathbb{R}_+$, it holds that $x_{\varepsilon,\delta} > 0$. Moreover,

$$\gamma_{\varepsilon,\delta} = \Phi_{\varepsilon,\delta}(x_{\varepsilon,\delta}, x'_{\varepsilon,\delta}) \geq \Phi_{\varepsilon,\delta}(x_0, x'_0) \geq \theta - 2\delta x_0^2 \geq \frac{\theta}{2}, \tag{36}$$

for any $\delta \leq \delta_0 = \frac{\theta}{4x_0^2}$. Obviously, for $\delta \leq \delta_0$ fixed, $\left(\gamma_{\varepsilon,\delta}\right)_\varepsilon$ is increasing and

$$\gamma_{2\varepsilon,\delta} \geq \gamma_{\varepsilon,\delta} + \frac{1}{4\varepsilon}\left(x_{\varepsilon,\delta} - x'_{\varepsilon,\delta}\right)^2.$$

Therefore,

$$\lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon}\left(x_{\varepsilon,\delta} - x'_{\varepsilon,\delta}\right)^2 = 0.$$

If, for all $\varepsilon > 0$ (or, at least for some arbitrary sequence ε_n such that $\varepsilon_n \to 0$ when $n \to \infty$) $x'_{\varepsilon,\delta} = 0$, then $\lim_{\varepsilon \searrow 0} x_{\varepsilon,\delta} = 0$, and, by taking the upper limit when $\varepsilon \to 0$ in (35), we get

$$\frac{\theta}{2} \leq U(0) - V(0) \leq 0,$$

which contradicts the assumption $\theta > 0$. We deduce that, for $\varepsilon > 0$ small enough, $x_{\varepsilon,\delta}$ and $x'_{\varepsilon,\delta}$ are strictly positive. We consider the test function

$$\varphi(x) = V(x'_{\varepsilon,\delta}) + \frac{1}{2\varepsilon}\left(x - x'_{\varepsilon,\delta}\right)^2 + \delta(x^2 + \left(x'_{\varepsilon,\delta}\right)^2), \text{ for } x \in \mathbb{R}_+^*,$$

such that $U - \varphi$ has a maximum point at $x_{\varepsilon,\delta}$. We write the variational inequality and use the previous Lemma to get

$$\max\left\{H(x_{\varepsilon,\delta}, U, \varphi'(x_{\varepsilon,\delta})), 1 - \varphi'(x_{\varepsilon,\delta})\right\} \geq 0. \tag{37}$$

In a similar way we have

$$\max\left\{H(x'_{\varepsilon,\delta}, V, \psi'(x'_{\varepsilon,\delta})), 1 - \psi'(x'_{\varepsilon,\delta})\right\} \leq 0, \tag{38}$$

where

$$\psi(x') = U(x_{\varepsilon,\delta}) - \frac{1}{2\varepsilon}(x_{\varepsilon,\delta} - x')^2 - \delta(x_{\varepsilon,\delta}^2 + (x')^2), \text{ for all } x' \in \mathbb{R}_+^*.$$

(a) We suppose that

$$H(x_{\varepsilon,\delta}, U, \varphi'(x_{\varepsilon,\delta})) \geq H(x'_{\varepsilon,\delta}, V, \psi'(x'_{\varepsilon,\delta})). \tag{39}$$

Then

$$
\begin{aligned}
0 \leq \sup_{u \geq \underline{u}} \{ & -r\left(U(x_{\varepsilon,\delta}) - V(x'_{\varepsilon,\delta})\right) \\
& + ap(u)\left[x_{\varepsilon,\delta}\varphi'(x_{\varepsilon,\delta}) - x'_{\varepsilon,\delta}\psi'(x'_{\varepsilon,\delta})\right] \\
& + \int_B \left(U(x_{\varepsilon,\delta} - f(ax_{\varepsilon,\delta}, y \wedge u)) - V(x'_{\varepsilon,\delta} - f(ax'_{\varepsilon,\delta}, y \wedge u))\right)\pi(dy) \\
& + \int_B \left(V(x'_{\varepsilon,\delta}) - U(x_{\varepsilon,\delta})\right)\pi(dy)\}. \qquad (40)
\end{aligned}
$$

We use $\Phi(x_{\varepsilon,\delta}; x'_{\varepsilon,\delta}) \geq \Phi(x_{\varepsilon,\delta} - f(x_{\varepsilon,\delta}, y \wedge u), x'_{\varepsilon,\delta} - f(x'_{\varepsilon,\delta}, y \wedge u))$ to get

$$
\begin{aligned}
& U(x_{\varepsilon,\delta} - f(ax_{\varepsilon,\delta}, y \wedge u)) - V(x'_{\varepsilon,\delta} - f(ax'_{\varepsilon,\delta}, y \wedge u)) \\
\leq & U(x_{\varepsilon,\delta}) - V(x'_{\varepsilon,\delta}) + \frac{1}{2\varepsilon}\left(x_{\varepsilon,\delta} - f(ax_{\varepsilon,\delta}, y \wedge u) - x'_{\varepsilon,\delta} + f(ax'_{\varepsilon,\delta}, y \wedge u)\right)^2 \\
& -\frac{1}{2\varepsilon}\left(x_{\varepsilon,\delta} - x'_{\varepsilon,\delta}\right)^2 \\
& +\delta\left((x_{\varepsilon,\delta} - f(ax_{\varepsilon,\delta}, y \wedge u))^2 - x^2_{\varepsilon,\delta} + \left(x'_{\varepsilon,\delta} - f(ax'_{\varepsilon,\delta}, y \wedge u)\right)^2 - \left(x'_{\varepsilon,\delta}\right)^2\right) \\
\leq & U(x_{\varepsilon,\delta}) - V(x'_{\varepsilon,\delta}),
\end{aligned}
$$

and, returning to (40), we have

$$
\begin{aligned}
0 \leq & \sup_{u \geq \underline{u}} \left\{-r\left(U(x_{\varepsilon,\delta}) - V(x'_{\varepsilon,\delta})\right) + 2ap(u)\left(\frac{\left(x_{\varepsilon,\delta} - x'_{\varepsilon,\delta}\right)^2}{2\varepsilon} + \delta\left(x^2_{\varepsilon,\delta} + \left(x'_{\varepsilon,\delta}\right)^2\right)\right)\right\} \\
\leq & \sup_{u \geq \underline{u}} (2ap(u) - r) \times \left(U(x_{\varepsilon,\delta}) - V(x'_{\varepsilon,\delta})\right).
\end{aligned}
$$

Recall that $\sup_{u \geq \underline{u}} 2ap(u) \leq \frac{2(1+k_1)\beta}{\zeta_0} < r$ (see (A2)). Thus, it follows that $\gamma_{\varepsilon,\delta} < 0$ which contradicts (36).

(b) If (39) does not hold, we use (37) and (38) and we must have

$$
1 - \varphi'(x_{\varepsilon,\delta}) \geq 0 \geq 1 - \psi'(x'_{\varepsilon,\delta}), \qquad (41)
$$

thus

$$
\frac{1}{\varepsilon}\left(x_{\varepsilon,\delta} - x'_{\varepsilon,\delta}\right) - 2\delta x'_{\varepsilon,\delta} \geq \frac{1}{\varepsilon}\left(x_{\varepsilon,\delta} - x'_{\varepsilon,\delta}\right) + 2\delta x_{\varepsilon,\delta}.
$$

We deduce that $x'_{\varepsilon,\delta} = x_{\varepsilon,\delta} = 0$ and get a contradiction. The proof of the comparison result is now complete.

6 Numerical results

We now turn our attention to some particular case and observe the optimal retention process by means of numerical simulation. We have seen that, for the collective risk model introduced in [9], a single insurance contract is considered and, of course, the risk is given for this one contract. A possible way to extend this model is to suppose that the risk concerns all contracts (or at least a percentage). We assume that the claims have constant intensity δ and the random measure μ is associated with some Poisson process of constant intensity $\pi(dy) = \beta G_\delta(dy)$, where G_δ corresponds to the Dirac mass. Moreover, the function f is given by $f(x,y) = \rho xy$, with $0 < \rho \leq 1$ (that is only some ρ part of the total contracts is subject to claims). In this case, the minimal retention level needed to cover expenditures is given explicitly by $\underline{u} = \frac{(k_2-k_1)\delta}{k_2}$ and $p(u) = (k_1 - k_2)\beta\rho\delta + (1+k_2)\beta\delta u$, for all $\frac{(k_2-k_1)\delta}{k_2} \leq u \leq \delta$.

Under the above assumptions, Eq. (5) reads

$$X_t^{x,u,L} = x + a\int_0^t X_s^{x,u,L} p(u_s)ds - \rho a \int_0^t X_{s-}^{x,u,L} u_s dN_s - \int_0^t dL_s. \tag{42}$$

Theorem 10 states that the maximized expected discounted dividends is the unique viscosity solution for the Hamilton-Jacobi-Bellman variational inequality

$$\begin{cases} \max\{H(x,V,V'(x)), 1 - V'(x)\} = 0 \text{ in } \mathbb{R}_+^*, \\ V(0) = 0, \end{cases} \tag{43}$$

where

$$H(x,V,q) = \sup_{\frac{(k_2-k_1)\delta}{k_2} \leq u \leq \delta} \{-rV(x) + axp(u)q + \beta[V(x - a\rho xu) - V(x)]\}.$$

The standard procedure in order to apply numerical arguments is to obtain a bounded space. Thus, we write the previous equation on $[0,1)$ by taking $y = \frac{x}{x+1}$ and $\psi(y) = V(x)$. This leads to the following HJB equation

$$\begin{cases} \max\left\{G(y,\psi,\psi'(y)), 1 - (1-y)^2\psi'(y)\right\} = 0 \text{ in } [0,1), \\ \psi(0) = 0, \end{cases} \tag{44}$$

where

$$G(y,\psi,q) =$$
$$\sup_{\frac{(k_2-k_1)\delta}{k_2} \leq u \leq \delta} \left\{-r\psi(y) + ap(u)y(1-y)q + \beta\left[\psi\left(\frac{y(1-a\rho u) - a\rho u}{a\rho uy + 1 - a\rho u}\right) - \psi(y)\right]\right\}.$$

As in Mnif, Sulem (2005), the approximate solution of Eq. (44) is computed with the help of finite difference approximations and the policy iteration algorithm.

We consider two particular cases: the first one illustrates the natural framework in which the reinsurance company perceives a relative safety loading greater than that of the insurer, while the second example assumes the opposite. The data set we use is given in the following table

	k_1	k_2	δ	r	β	ρ
Fig 1	2	0.25	1	0.07	0.0011	10%
Fig 2	2	0.19	1	0.07	0.0011	10%

For the first framework, the optimal retention level turns out to be maximal as shown by Fig 1.

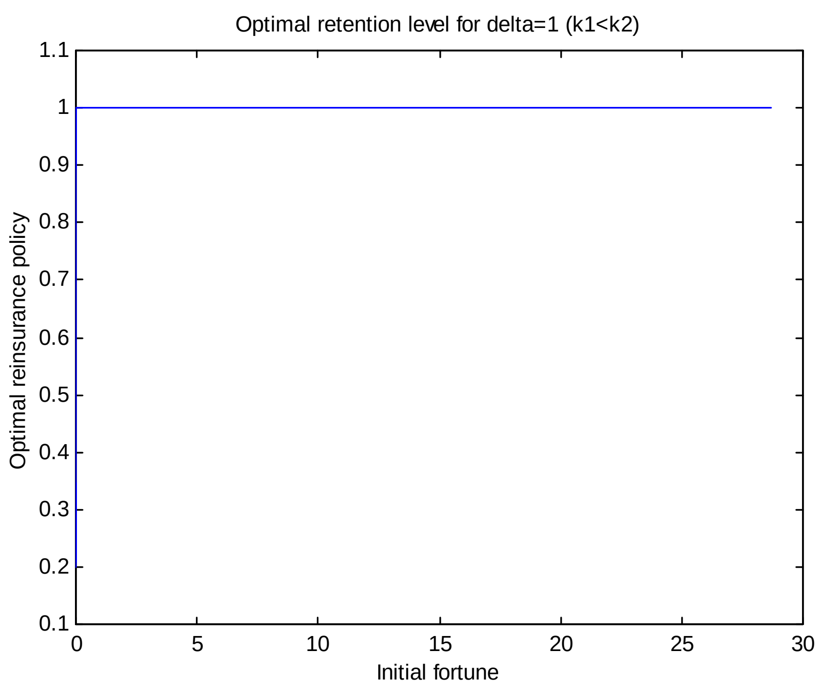

Fig 1. Optimal retention level for $\delta = 1$, $k_1 = 0.2$, $k_2 = 0.25$

As can be expected in the second case, if the initial reserve is great enough, then the direct insurer should play the safety card in order to maximize expected discounted dividends. Indeed, since the relative safety loadings guarantee a proportional steady income to the insurer, the optimal retention level is null (see Fig 2).

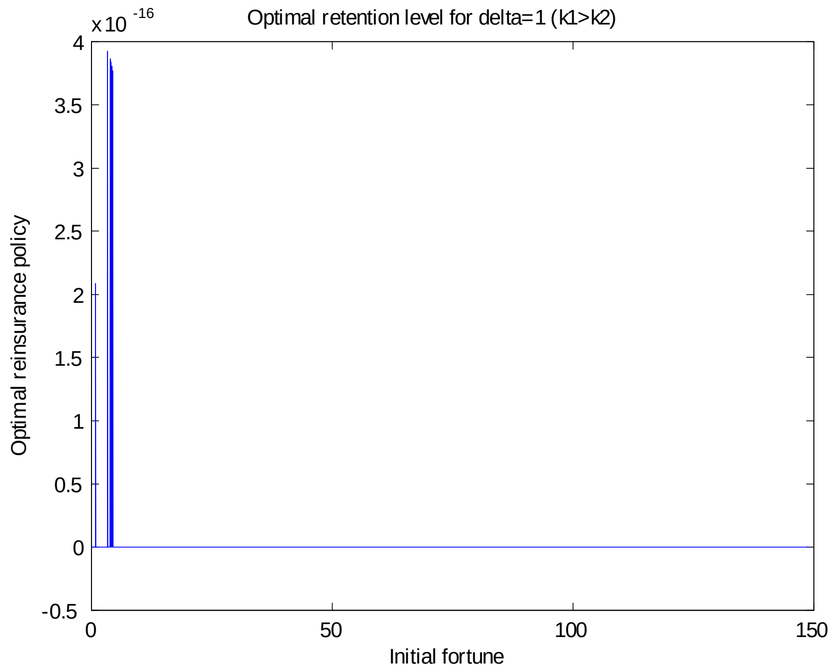

Fig 2. Optimal retention level for $\delta = 1$, $k_1 = 0.2$, $k_2 = 0.19$

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